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Guaranteed Cost Controller Design for Multirate Systems

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Abstract: This paper proposes a design method of a guaranteed cost controller for multirate systems. A class of a minimal order observer is used to design the robust controller via linear matrix inequality technique. A numerical example is given to illustrate the proposed method.

Keywords: guaranteed cost control, multirate systems, a minimal order observer, LMI

1. INTRODUCTION

Instability and bad performance can occur on a controlled system with uncertainties. Therefore, considerable interests have been attracted to studies of robust controller design in recent decades. Moreover, it is desirable to design a controller which not only achieves the stability of the uncertain system but also guarantees an adequate level of performance. One of the approaches to solve this problem is a guaranteed cost control method [1].

Multirate sampling schemes have long been the focus of interest by many control designers [2], [3], [4], [5]. This paper proposes a design method of a guaranteed cost controller for multirate systems. The controller is based on a state-space model and the state variable is estimated by a minimal order multirate observer. The design problem is expressed by matrix inequalities and solved by an iterative algorithm of a linear matrix inequality technique.

2. PROBLEM STATEMENT

Consider a discrete-time uncertain system in the form

$$\begin{aligned} \boldsymbol{x}(k+1) &= (A_0 + \Delta A_0)\boldsymbol{x}(k) + (B_0 + \Delta B_0)\boldsymbol{u}(k) \\ \boldsymbol{y}(k) &= C\boldsymbol{x}(k) \end{aligned} \tag{1}$$

Assume the output sampling period is greater than that of the input and only $\boldsymbol{y}(kNT_s)$ is available, where N is a positive integer greater than one and T_s denotes a sampling period. Then, the general form of (1) can be written as follows

$$\begin{aligned} \boldsymbol{x}(k+N) &= (A + \Delta A)\boldsymbol{x}(k) + (B + \Delta B)\boldsymbol{v}(k) \\ \boldsymbol{y}(k) &= C\boldsymbol{x}(k) \end{aligned} \tag{2}$$

where

$$\begin{aligned} A &= A_0^N,\ \Delta A = (A_0 + \Delta A_0)^N - A_0^N, \\ B &= [B_0, A_0B_0, \cdots, A_0^{N-1}B_0], \\ \Delta B &= [\Delta B_0, (A_0 + \Delta A_0)(B_0 + \Delta B_0) - A_0B_0, \cdots, \\ &\quad (A_0 + \Delta A_0)^{N-1}(B_0 + \Delta B_0) - A_0^{N-1}B_0], \\ \boldsymbol{v}(k) &= [\boldsymbol{u}^T(k+N-1), \boldsymbol{u}^T(k+N-2), \cdots, \boldsymbol{u}^T(k)]^T. \end{aligned}$$

Matrices A_0, and B_0 are known constant real-valued matrices with appropriate dimensions, and C is restricted to the form of $C = [O\ \ I_m]$.

We assume that the parameter uncertainties $\Delta A_0(k)$ and $\Delta B_0(k)$ satisfy the following relations

$$\Delta A_0(k) = D_AF_A(k)E_A,\ \Delta B_0(k) = D_BF_B(k)E_B \tag{3}$$

where $F_A(k)$ and $F_B(k)$ are unknown time-varying and deterministic matrices satisfying

$$F_A^T(k)F_A(k) \le I,\ F_B^T(k)F_B(k) \le I \tag{4}$$

and D_A, E_A are constant real-valued known matrices with appropriate dimensions.

It is also assumed that the initial state variable $\boldsymbol{x}(0)$ is unknown, but their mean and covariance are known, respectively as

$$E[\boldsymbol{x}(0)] = \boldsymbol{m}_0 \tag{5}$$

$$E\left[(\boldsymbol{x}(0) - \boldsymbol{m}_0)(\boldsymbol{x}(0) - \boldsymbol{m}_0)^T\right] = \Sigma_0 > O \tag{6}$$

where $E[\cdot]$ denotes the expected value operator.

The problem considered here is to design a minimal order observer

$$\boldsymbol{z}(k+N) = D\boldsymbol{z}(k) + E\boldsymbol{y}(k) + \sum_{i=0}^{N-1} H_i\boldsymbol{u}(k+i) \tag{7}$$

$$\hat{\boldsymbol{x}}(k) = P\boldsymbol{z}(k) + W\boldsymbol{y}(k) \tag{8}$$

and a controller

$$\boldsymbol{v}(k) = K\hat{\boldsymbol{x}}(k) \tag{9}$$

with

$$K = \begin{bmatrix} K_{N-1} \\ \vdots \\ K_1 \\ K_0 \end{bmatrix},\ A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},\ D = A_{11} + LA_{21},$$

$$H_i = TA^{N-1-i}B,\ TA - DT = EC,$$

$$T = \begin{bmatrix} I_{n-m} & L \end{bmatrix},\ PT + WC = I_n,\ P = \begin{bmatrix} I_{n-m} & O \end{bmatrix}^T$$

so as to achieve an upper bound on the following quadratic performance index

$$E[J] = E\left[\sum_{l=0} \{\boldsymbol{x}^T(Nl)Q\boldsymbol{x}(Nl) + \boldsymbol{v}^T(Nl)R\boldsymbol{v}(Nl)\}\right] \tag{10}$$

associated with the multirate systems (2), where Q and R are given symmetric positive-definite matrices.

3. MAIN RESULTS

Attention of this study is restricted to $N = 2$, $\Delta B_0 = 0$ for simplicity of description. The main result of this study is given by Theorem 1.

Theorem 1. If the following matrix inequalities optimization problem; $\min\{\gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4\}$ subject to

$$\begin{bmatrix} -S_1+Q & 0 & 0 & K^TR & (A+BK)^T \\ 0 & -S_2 & D^TS_2 & (KP)^TR & (BKP)^T \\ 0 & * & -S_2 & 0 & 0 \\ * & * & 0 & -R & 0 \\ * & * & 0 & 0 & -X \end{bmatrix} < 0 \quad (11)$$

$$\begin{bmatrix} \Psi & E_A^T & \Psi_1 & \Psi_2 & \Psi_3 & \Psi_4 & \Psi_5 & \Psi_6 & \Psi_7 & \Psi_8 & \Psi_9 \\ * & M_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & M_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & M_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & M_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & M_4 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & M_5 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & M_6 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_7 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_8 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_9 \end{bmatrix} < 0 \quad (12)$$

$$\begin{bmatrix} -X_1 & S_1A_0D_A \\ * & -\delta_4 I \end{bmatrix} < 0, \begin{bmatrix} -X_2 & S_1D_A \\ * & -\delta_5 I \end{bmatrix} < 0,$$
$$\begin{bmatrix} -X_3 & S_1D_A \\ * & -\delta_6 I \end{bmatrix} < 0, \begin{bmatrix} -X_4 & S_1D_A \\ * & -\delta_{11} I \end{bmatrix} < 0,$$
$$\begin{bmatrix} -X_5 & S_1D_A \\ * & -\delta_{28inv} I \end{bmatrix} < 0, \begin{bmatrix} -Y_1 & S_1A_0D_A \\ * & -\delta_{13} I \end{bmatrix} < 0,$$
$$\begin{bmatrix} -Y_2 & S_1D_A \\ * & -\delta_{14} I \end{bmatrix} < 0, \begin{bmatrix} -Y_3 & S_1D_A \\ * & -\delta_{15} I \end{bmatrix} < 0,$$
$$\begin{bmatrix} -Y_4 & S_1D_A \\ * & -\delta_{19} I \end{bmatrix} < 0, \begin{bmatrix} -Y_5 & S_1D_A \\ * & -\delta_{31inv} I \end{bmatrix} < 0, \quad (13)$$

$$\sum_{k=1}^{n} \boldsymbol{e}_{nk}^T \Theta_0 \boldsymbol{e}_{nk} < \gamma_0, \quad \sum_{k=1}^{m} \boldsymbol{e}_{mk}^T \Theta_1 \boldsymbol{e}_{mk} < \gamma_1$$
$$\sum_{k=1}^{m} \boldsymbol{e}_{mk}^T \Theta_2 \boldsymbol{e}_{mk} < \gamma_2, \quad \sum_{k=1}^{m} \boldsymbol{e}_{mk}^T \Theta_3 \boldsymbol{e}_{mk} < \gamma_3 \quad (14)$$

$$\begin{bmatrix} -\gamma_4 & \boldsymbol{w}_1^T Y^T & \boldsymbol{w}_2^T Y^T & \cdots & \boldsymbol{w}_m^T Y^T \\ Y\boldsymbol{w}_1 & -S_2 & & & \vdots \\ Y\boldsymbol{w}_2 & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ Y\boldsymbol{w}_m & \cdots & \cdots & \cdots & -S_2 \end{bmatrix} < 0 \quad (15)$$

where

$$\begin{aligned} \Psi = {} & (\delta_1 + \delta_3\mu_3 + \mu_a + (\delta_c)_{inv}\epsilon_c + (\delta_e)_{inv}\epsilon_e + \mu_f \\ & +\delta_4 + \delta_6\mu_6 + \delta_9\mu_9 + \mu_g + (\delta_i)_{inv}\mu_i \\ & +(\delta_l)_{inv}\mu_l + \mu_m + \delta_{13} + \delta_{15}\mu_{15} + \delta_{16}\mu_{16} \\ & +\delta_{18}\mu_{18} + \delta_{21} + \delta_{23}\mu_{23} + \delta_{24}\mu_{24} + \delta_{26}\mu_{26}) \\ & E_A^TE_A + (\delta_2 + (\delta_b)_{inv} + \mu_d + \delta_5 + (\delta_h)_{inv} \\ & +\delta_j + \delta_{14} + \delta_{17}\mu_{17} + \delta_{22} + \delta_{25}\mu_{25}) \\ & (E_AA_0)^TE_AA_0, \end{aligned}$$

$$\begin{aligned} & \Psi_1 = A^TS_1A_0D_A, \ \Psi_2 = A^TS_1D_A, \\ & \Psi_3 = (E_AA_0)^T, \ \Psi_4 = (BK)^T, \ \Psi_5 = (E_AB_0K_0)^T, \\ & \Psi_6 = (BKP)^T, \ \Psi_7 = (E_AB_0K_0P)^T, \\ & \Psi_8 = Z_CA_0D_A, \ \Psi_9 = Z_CD_A, \\ & M_0 = -\mathrm{diag}\{(\delta_b)_{inv}(\mu_b)_{inv}, (\delta_c)_{inv}(\mu_c)_{inv}, \\ & \qquad \delta_{7inv}\mu_{7inv}, (\delta_h)_{inv}(\mu_h)_{inv}, (\delta_i)_{inv}(\mu_i)_{inv}\}, \\ & M_1 = -\delta_1, \ M_2 = -\mathrm{diag}\{\delta_2, \delta_3, \delta_{10}, \delta_{27inv}\}, \\ & M_3 = -\mathrm{diag}\{(\delta_e)_{inv}(\mu_e)_{inv}, \delta_{8inv}\mu_{8inv}, (\delta_l)_{inv}\mu_l\}, \\ & M_4 = -\mathrm{diag}\{X_{1inv}, X_{2inv}, X_{3inv}, X_{4inv}, X_{5inv}\}, \\ & M_5 = -\mathrm{diag}\{\delta_7, \delta_8, \delta_9\epsilon_{9inv}, \delta_{10inv}, \delta_{11inv}, \\ & \qquad \delta_{12inv}\mu_{12inv}, \delta_{19inv}, \delta_{20inv}\mu_{20inv}\}, \\ & M_6 = -\mathrm{diag}\{Y_{1inv}, Y_{2inv}, Y_{3inv}, Y_{4inv}, Y_{5inv}\}, \\ & M_7 = -\mathrm{diag}\{\delta_{16}, \delta_{17}, \delta_{18}\epsilon_{18inv}, \delta_{20}, \delta_{24}, \delta_{25}, \\ & \qquad \delta_{26inv}\epsilon_{26inv}, \delta_{27}, \delta_{28}, \delta_{29}, \mu_{30inv}, \delta_{31}, \mu_{32}\}, \\ & M_8 = -\delta_{21}, \ M_9 = -\mathrm{diag}\{\delta_{22}, \delta_{23}, \delta_{29inv}\}, \\ & Y = S_2L, \ Z = [S_2 \ \ Y], \\ & \Theta_0 = \frac{1}{2}(S_1(\Sigma_0 + \boldsymbol{m}_0\boldsymbol{m}_0^T) + (\Sigma_0 + \boldsymbol{m}_0\boldsymbol{m}_0^T)^TS_1), \\ & \Theta_1 = \frac{1}{2}(S_2\Sigma_{11} + \Sigma_{11}S_2), \ \Theta_2 = \frac{1}{2}(Y\Sigma_{21} + \Sigma_{21}^TY^T), \\ & \Theta_3 = \frac{1}{2}(Y^T\Sigma_{12} + \Sigma_{12}^TY), \ \Sigma_{22}^{1/2} = [\boldsymbol{w}_1, \boldsymbol{w}_2, \cdots, \boldsymbol{w}_m], \\ & \Sigma_0 = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}, \ \boldsymbol{e}_{ik} = \begin{bmatrix} 0_{k-1}^T & 1 & 0_{i-k}^T \end{bmatrix}^T, \end{aligned}$$

has a set of solutions $S_1 > 0$, $X > 0$, $X_l > 0$, $(X_l)_{inv} > 0$, $Y_l > 0$, $(Y_l)_{inv} > 0$, where $l = 1, \cdots, 5$, Y, Z, γ_0, γ_1, γ_2, γ_3, γ_4 and positive scalars δ_1, δ_2, δ_3, and so on which satisfy the inverse relation such as $X = S_1^{-1}$, $(X_l)_{inv} = X_l^{-1}$ and so on, then the minimal order observer-based control law (7)-(9) is a guaranteed cost controller which gives the minimum expected value of the guaranteed cost

$$\begin{aligned} E[J] &= E\left[\boldsymbol{x}^T(0)S_1\boldsymbol{x}(0) + \boldsymbol{\xi}^T(0)S_2\boldsymbol{\xi}(0)\right] \\ &= \min\{\gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4\} \end{aligned} \quad (16)$$

where $\boldsymbol{\xi}(k) = \boldsymbol{z}(k) - T\boldsymbol{x}(k)$ is the estimated error of the minimal order observer.

Remark 1: Since (11)-(13) have a constraint of the inverse relations and nonlinear terms, an iterative algorithm via LMI approach is introduced to solve [6].

Before giving a proof of theorem 1, a key lemma is introduced.

Lemma 1. [7] Given matrices D and E of appropriate dimensions, and $F(k)$ be a matrix function satisfying $F(k)^TF(k) \leq I$, then for any $\alpha > 0$, the following inequality holds

$$DF(k)E + E^TF(k)^TE^T \leq \alpha DD^T + \alpha^{-1}E^TE.$$

Proof of Theorem 1.
Equations (2) and (7)-(9) yield the closed-loop system

$$\begin{bmatrix} \boldsymbol{x}(k+2) \\ \boldsymbol{\xi}(k+2) \end{bmatrix} = \begin{bmatrix} \Phi_1 & \Phi_2 \\ \Phi_3 & \Phi_4 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(k) \\ \boldsymbol{\xi}(k) \end{bmatrix} \quad (17)$$

where

$$\Phi_1 = A + \Delta A(k) + BK,\ \Phi_2 = (B + \Delta B(k))KP,$$
$$\Phi_3 = -T\Delta A(k),\ \Phi_4 = D - T\Delta B(k)KP.$$

Consider a candidate of Lyapunov function as

$$V(k) = \boldsymbol{x}^T(k)S_1\boldsymbol{x}(k) + \boldsymbol{\xi}^T S_2\boldsymbol{\xi}(k) \tag{18}$$

then,

$$\begin{aligned}\Delta V(k) &= V(k+2) - V(k)\\ &= \begin{bmatrix}\boldsymbol{x}^T(k)\\ \boldsymbol{\xi}^T(k)\end{bmatrix}\begin{bmatrix}\Lambda_1 & \Lambda_2\\ * & \Lambda_3\end{bmatrix}\begin{bmatrix}\boldsymbol{x}(k)\\ \boldsymbol{\xi}(k)\end{bmatrix}\end{aligned} \tag{19}$$

where

$$\begin{aligned}\Lambda_1 &= (A + \Delta A(k) + (B + \Delta B(k))K)^T S_1\\ &\quad (A + \Delta A(k) + (B + \Delta B(k))K)\\ &\quad +(-T\Delta A(k))^T S_2(-T\Delta A(k)) - S_1,\\ \Lambda_2 &= (A + \Delta A(k) + (B + \Delta B(k))K)^T S_1\\ &\quad (B + \Delta B(k))KP + (-T\Delta A(k))^T S_2\\ &\quad (D - T\Delta B(k)KP),\\ \Lambda_3 &= (D - T\Delta B(k)KP)^T S_2(D - T\Delta B(k)KP)\\ &\quad +((B + \Delta B(k))KP)^T S_1(B + \Delta B(k))KP\\ &\quad -S_2.\end{aligned}$$

By introducing $\bar{\boldsymbol{z}}(k)$ and Ω, the equation (19) can be written as

$$\begin{aligned}\Delta V(k) &= \bar{\boldsymbol{z}}^T(k)\Omega\bar{\boldsymbol{z}}(k)\\ &\quad -\{\boldsymbol{x}^T(k)Q\boldsymbol{x}(k) + \boldsymbol{v}^T(k)R\boldsymbol{v}(k)\}\end{aligned} \tag{20}$$

where

$$\bar{\boldsymbol{z}}(k) = \begin{bmatrix}\boldsymbol{x}(k)\\ \boldsymbol{\xi}(k)\end{bmatrix},\ \Omega = \begin{bmatrix}\bar{\Lambda}_1 & \bar{\Lambda}_2\\ * & \bar{\Lambda}_3\end{bmatrix},$$

$$\begin{aligned}\bar{\Lambda}_1 &= (A + \Delta A(k) + (B + \Delta B(k))K)^T S_1\\ &\quad (A + \Delta A(k) + (B + \Delta B(k))K)\\ &\quad +(-T\Delta A(k))^T S_2(-T\Delta A(k)) - S_1\\ &\quad +Q + K^T RK\\ \bar{\Lambda}_2 &= (A + \Delta A(k) + (B + \Delta B(k))K)^T S_1\\ &\quad (B + \Delta B(k))KP + (-T\Delta A(k))^T S_2\\ &\quad (D - T\Delta B(k)KP) + K^T RKP\\ \bar{\Lambda}_3 &= (D - T\Delta B(k)KP)^T S_2(D - T\Delta B(k)KP)\\ &\quad +((B + \Delta B(k))KP)^T S_1(B + \Delta B(k))KP\\ &\quad -S_2 + (KP)^T RKP.\end{aligned}$$

Under the condition

$$\Omega < 0 \tag{21}$$

the equation (20) leads to

$$\Delta V(k) < -\{\boldsymbol{x}^T(k)Q\boldsymbol{x}(k) + \boldsymbol{v}^T(k)R\boldsymbol{v}(k)\} < 0 \tag{22}$$

for any $\boldsymbol{x}(k) \neq \mathbf{0}$ and the closed-loop system is asymptotically stable.

Here, the condition (21) is investigated below. By applying Lemma 1 to (21), the parameter uncertainties can be written down

$$\begin{aligned}&2\boldsymbol{x}^T(k)\Delta A^T(k)S_1A\boldsymbol{x}(k)\\ &\leq \boldsymbol{x}^T(k)\{(\delta_1 + \delta_3\mu_3)E_A^T E_A + \delta_1^{-1}A^T S_1A_0D_A\\ &\quad (S_1A_0D_A)^T A + \delta_2(E_AA_0)^T E_AA_0\\ &\quad + (\delta_2^{-1} + \delta_3^{-1})A^T S_1D_AD_A^T S_1A\}\boldsymbol{x}(k)\end{aligned} \tag{23}$$

$$\begin{aligned}&\boldsymbol{x}^T(k)\Delta A^T(k)S_1\Delta A\boldsymbol{x}(k)\\ &\leq \boldsymbol{x}^T(k)\{(\delta_b^{-1} + \mu_d + \delta_e\mu_e)(E_AA_0)^T E_AA_0\\ &\quad + (\mu_a + \delta_b\mu_b + \delta_c\mu_c + \delta_c^{-1}\epsilon_c + \delta_e\mu_e + \mu_f)\\ &\quad E_A^T E_A\}\boldsymbol{x}(k)\end{aligned} \tag{24}$$

$$\begin{aligned}&2\boldsymbol{x}^T(k)\Delta A^T(k)S_1BK\boldsymbol{x}(k)\\ &\leq \boldsymbol{x}^T(k)\{(\delta_4 + \delta_6\mu_6)E_A^T E_A + \delta_5(E_AA_0)^T E_AA_0\\ &\quad + \delta_4^{-1}K^T B^T S_1A_0D_A(S_1A_0D_A)^T BK\\ &\quad + (\delta_5^{-1} + \delta_6^{-1})K^T B^T S_1D_A(S_1D_A)^T BK\}\boldsymbol{x}(k)\end{aligned} \tag{25}$$

$$\begin{aligned}&2\boldsymbol{x}^T(k)\Delta A^T(k)S_1\Delta BK\boldsymbol{x}(k)\\ &\leq \boldsymbol{x}^T(k)\{(\delta_7\mu_7 + \delta_9\mu_9)E_A^T E_A\\ &\quad + \delta_8\mu_8(E_AA_0)^T E_AA_0 + (\delta_7^{-1} + \delta_8^{-1} + \delta_9^{-1}\epsilon_9)\\ &\quad (E_AB_0K_0)^T E_AB_0K_0\}\boldsymbol{x}(k)\end{aligned} \tag{26}$$

$$\begin{aligned}&2\boldsymbol{x}^T(k)K^T\Delta B^T(k)S_1A\boldsymbol{x}(k)\\ &\leq \boldsymbol{x}^T(k)\{\delta_{10}(E_AB_0K_0)^T E_AB_0K_0\\ &\quad + \delta_{10}^{-1}A^T S_1D_AD_A^T S_1A\}\boldsymbol{x}(k)\end{aligned} \tag{27}$$

$$\begin{aligned}&2\boldsymbol{x}^T(k)K^T\Delta B^T(k)S_1BK\boldsymbol{x}(k)\\ &\leq \boldsymbol{x}^T(k)\{\delta_{11}(E_AB_0K_0)^T E_AB_0K_0\\ &\quad + \delta_{11}^{-1}(BK)^T S_1D_AD_A^T S_1BK\}\boldsymbol{x}(k)\end{aligned} \tag{28}$$

$$\begin{aligned}&2\boldsymbol{x}^T(k)K^T\Delta B^T(k)S_1\Delta B(k)K\boldsymbol{x}(k)\\ &\leq \boldsymbol{x}^T(k)\{\delta_{12}\mu_{12}(E_AB_0K_0)^T E_AB_0K_0\}\boldsymbol{x}(k)\end{aligned} \tag{29}$$

$$\begin{aligned}&\boldsymbol{x}^T(k)\Delta A^T(k)T^T S_2T\Delta A(k)\boldsymbol{x}(k)\\ &\leq \boldsymbol{x}^T(k)\{(\mu_g + \delta_h\mu_h + \delta_i\mu_i + \delta_i^{-1}\epsilon_i + \delta_l^{-1}\epsilon_l + \mu_m)\\ &\quad E_A^T E_A + (\delta_h^{-1} + \delta_j + \delta_l\mu_l)E_A^T E_A\}\boldsymbol{x}(k)\end{aligned} \tag{30}$$

$$\begin{aligned}&2\boldsymbol{x}^T(k)\Delta A^T(k)S_1BKP\boldsymbol{\xi}(k)\\ &\leq \boldsymbol{x}^T(k)\{(\delta_{13} + \delta_{15}\mu_{15})E_A^T E_A + \delta_{14}(E_AA_0)^T\\ &\quad E_AA_0\}\boldsymbol{x}(k) + \boldsymbol{\xi}^T(k)\{\delta_{13}^{-1}(E_AB_0K_0P)^T\\ &\quad S_1A_0D_A(S_1A_0D_A)^T E_AB_0K_0P + (\delta_{14}^{-1} + \delta_{15}^{-1})\\ &\quad (E_AB_0K_0P)^T S_1D_A(S_1D_A)^T E_AB_0K_0P\}\boldsymbol{\xi}(k)\end{aligned} \tag{31}$$

$$\begin{aligned}&2\boldsymbol{x}^T(k)\Delta A^T(k)S_1\Delta BKP\boldsymbol{\xi}(k)\\ &\leq \boldsymbol{x}^T(k)\{(\delta_{16}\mu_{16} + \delta_{18}\mu_{18})E_A^T E_A\\ &\quad + \delta_{17}\mu_{17}(E_AA_0)^T E_AA_0\}\boldsymbol{x}(k)\\ &\quad + \boldsymbol{\xi}^T(k)\{(\delta_{16}^{-1} + \delta_{17}^{-1} + \delta_{18}^{-1}\epsilon_{18})(E_AB_0K_0P)^T\\ &\quad E_AB_0K_0P\}\boldsymbol{\xi}(k)\end{aligned} \tag{32}$$

$$2\boldsymbol{x}^T(k)K^T\Delta B^T(k)S_1BKP\boldsymbol{\xi}(k) \le \boldsymbol{x}^T(k)\{\delta_{19}(E_AB_0K_0)^TE_AB_0K_0\}\boldsymbol{x}(k) + \boldsymbol{\xi}^T(k)\{\delta_{19}^{-1}(BKP)^TS_1D_AD_A^TS_1BKP\}\boldsymbol{\xi}(k) \quad (33)$$

$$2\boldsymbol{x}^T(k)K^T\Delta B^T(k)S_1\Delta BKP\boldsymbol{\xi}(k) \le \boldsymbol{x}^T(k)\{\delta_{20}\mu_{20}(E_AB_0K_0)^TE_AB_0K_0\}\boldsymbol{x}(k) + \boldsymbol{\xi}^T(k)\{\delta_{20}^{-1}(E_AB_0K_0P)^TE_AB_0K_0P\}\boldsymbol{\xi}(k) \quad (34)$$

$$-2\boldsymbol{x}^T(k)\Delta A^T(k)T^TS_2D\boldsymbol{\xi}(k) \le \boldsymbol{x}^T(k)\{(\delta_{21}+\delta_{23}\mu_{23})E_A^TE_A + \delta_{22}(E_AA_0)^TE_AA_0\}\boldsymbol{x}(k) + \boldsymbol{\xi}^T(k)\{\delta_{21}^{-1}Z_CA_0D_A(Z_CA_0D_A)^T + (\delta_{22}^{-1}+\delta_{23}^{-1})Z_CD_A(Z_CD_A)^T\}\boldsymbol{\xi}(k) \quad (35)$$

$$2\boldsymbol{x}^T(k)\Delta A^T(k)T^TS_2T\Delta B(k)KP\boldsymbol{\xi}(k) \le \boldsymbol{x}^T(k)\{(\delta_{24}\mu_{24}+\delta_{26}\mu_{26})E_A^TE_A + \delta_{25}\mu_{25}(E_AA_0)^TE_AA_0\}\boldsymbol{x}(k) + \boldsymbol{\xi}^T(k)\{(\delta_{24}^{-1}+\delta_{25}^{-1}+\delta_{26}^{-1}\epsilon_{26})(E_AB_0K_0P)^TE_AB_0K_0P\}\boldsymbol{\xi}(k) \quad (36)$$

$$2\boldsymbol{x}^T(k)A^TS_1\Delta B(k)KP\boldsymbol{\xi}(k) \le \boldsymbol{x}^T(k)\{\delta_{27}A^TS_1D_AD_A^TS_1A\}\boldsymbol{x}(k) + \boldsymbol{\xi}^T(k)\{\delta_{27}^{-1}(E_AB_0K_0P)^TE_AB_0K_0P\}\boldsymbol{\xi}(k) \quad (37)$$

$$2\boldsymbol{x}^T(k)K^TB^TS_1\Delta B(k)KP\boldsymbol{\xi}(k) \le \boldsymbol{x}^T(k)\{\delta_{28}(BK)^TS_1D_AD_A^TS_1BK\}\boldsymbol{x}(k) + \boldsymbol{\xi}^T(k)\{\delta_{28}^{-1}(E_AB_0K_0P)^TE_AB_0K_0P\}\boldsymbol{\xi}(k) \quad (38)$$

$$-2\boldsymbol{\xi}^T(k)D^TS_2T\Delta B(k)KP\boldsymbol{\xi}(k) \le \boldsymbol{\xi}^T(k)\{\delta_{29}Z_CD_A(Z_CD_A)^T + \delta_{29}^{-1}(E_AB_0K_0P)^TE_AB_0K_0P\}\boldsymbol{\xi}(k) \quad (39)$$

$$2\boldsymbol{\xi}^T(k)(T\Delta B(k)KP)^TS_2T\Delta B(k)KP\boldsymbol{\xi}(k) \le \boldsymbol{\xi}^T(k)\{\mu_{30}(E_AB_0K_0P)^TE_AB_0K_0P\}\boldsymbol{\xi}(k) \quad (40)$$

$$2\boldsymbol{\xi}^T(k)(BKP)^TS_1T\Delta B(k)KP\boldsymbol{\xi}(k) \le \boldsymbol{\xi}^T(k)\{\delta_{31}(BKP)^TS_1D_AD_A^TS_1BKP + \delta_{31}^{-1}(E_AB_0K_0P)^TE_AB_0K_0P\}\boldsymbol{\xi}(k) \quad (41)$$

$$2\boldsymbol{\xi}^T(k)(B(k)KP)^TS_1\Delta B(k)KP\boldsymbol{\xi}(k) \le \boldsymbol{\xi}^T(k)\{\mu_{32}(E_AB_0K_0P)^TE_AB_0K_0P\}\boldsymbol{\xi}(k) \quad (42)$$

Since there are some inequalities in (23)-(42) that can not be built to LMIs directly, we define new variables to solve these problems.

$$\frac{1}{\delta_4}S_1A_0D_A(S_1A_0D_A)^T \le X_1,\ \frac{1}{\delta_5}S_1D_A(S_1D_A)^T \le X_2,\ \frac{1}{\delta_5}S_1D_A(S_1D_A)^T \le X_3,\ \frac{1}{\delta_{11}}S_1D_A(S_1D_A)^T \le X_4,\ \delta_{28}S_1D_A(S_1D_A)^T \le X_5,\ \frac{1}{\delta_{13}}S_1A_0D_A(S_1A_0D_A)^T \le Y_1,\ \frac{1}{\delta_{14}}S_1D_A(S_1D_A)^T \le Y_2,\ \frac{1}{\delta_{15}}S_1D_A(S_1D_A)^T \le Y_3,\ \frac{1}{\delta_{19}}S_1D_A(S_1D_A)^T \le Y_4,\ \delta_{31}S_1D_A(S_1D_A)^T \le Y_5, \quad (43)$$

By using Schur complement, (23)-(42) lead to (12). Ignoring parameter uncertainties, Ω in (20) equivalent to

$$\begin{bmatrix} -S_1+Q & 0 & 0 & K^TR & (A+BK)^TS_1 \\ 0 & -S_2 & D^TS_2 & (KP)^TR & (BKP)^TS_1 \\ 0 & * & -S_2 & 0 & 0 \\ * & * & 0 & -R & 0 \\ * & * & 0 & 0 & -S_1 \end{bmatrix} < 0 \quad (44)$$

Pre- and post-multiplying (44) by diag$(I,\ I,\ I,\ I,\ S_1^{-1})$ on both sides, denoting $X = S_1^{-1}$ and using Schur Complement lead to (11). Further, summing (22) from 0 to ∞ yields

$$J = \sum_{k=2i,i=0} (\boldsymbol{x}^T(k)Q\boldsymbol{x}(k) + \boldsymbol{v}^T(k)R\boldsymbol{v}(k)) < \boldsymbol{x}^T(0)S_1\boldsymbol{x}(0) + \boldsymbol{\xi}^T(0)S_2\boldsymbol{\xi}(0) = J^* \quad (45)$$

where i is positive integer and J^* denotes the guaranteed cost. Here, we consider the optimal expected value of the guaranteed cost. It is calculated as

$$E[J^*] = \mathrm{tr}S_1E\left[\boldsymbol{x}(0)\boldsymbol{x}^T(0)\right] + \mathrm{tr}S_2E\left[\boldsymbol{\xi}(0)\boldsymbol{\xi}^T(0)\right] \quad (46)$$

A relation between mean and covariance of $\boldsymbol{x}(0)$ is given by

$$\Sigma_0 = E\left[\boldsymbol{x}(0)\boldsymbol{x}^T(0)\right] - \boldsymbol{m}_0\boldsymbol{m}_0^T \quad (47)$$

Substituting (47) into (46) results in

$$E[J^*] = \mathrm{tr}S_1(\Sigma_0 + \boldsymbol{m}_0\boldsymbol{m}_0^T) + \mathrm{tr}S_2E\left[(\boldsymbol{z}(0)-T\boldsymbol{x}(0))(\boldsymbol{z}(0)-T\boldsymbol{x}(0))^T\right] \quad (48)$$

Here, it is readily seen from (48) that

$$E\left[(\boldsymbol{z}(0)-T\boldsymbol{x}(0))(\boldsymbol{z}(0)-T\boldsymbol{x}(0))^T\right] = T\Sigma_0T^T + (\boldsymbol{z}(0)-T\boldsymbol{m}_0)(\boldsymbol{z}(0)-T\boldsymbol{m}_0)^T, \quad (49)$$

Next, consider positive scalars $\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4$, satisfying the following inequalities

$$\mathrm{tr}S_1(\Sigma_0 + \boldsymbol{m}_0\boldsymbol{m}_0^T) < \gamma_0 \quad (50)$$

$$\mathrm{tr}S_2\Sigma_{11} < \gamma_1 \quad (51)$$

$$\mathrm{tr}S_2L\Sigma_{21} < \gamma_2 \quad (52)$$

$$\mathrm{tr}S_2\Sigma_{12}L^T < \gamma_3 \quad (53)$$

$$\mathrm{tr}S_2L\Sigma_{22}L^T < \gamma_4 \quad (54)$$

Minimizing $\gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ results in giving min $E[J^*]$. By recalling tr(AB) = tr(BA), (50)-(53) lead to (14). Next, by denoting $\Sigma_{22}^{1/2} = [\boldsymbol{w}_1, \boldsymbol{w}_2, \cdots, \boldsymbol{w}_m]$, equation (54) is calculated as

$$\begin{aligned}\text{tr}S_2 L\Sigma_{22}L^T &= \boldsymbol{w}_1^T Y^T S_2^{-1} Y\boldsymbol{w}_1 + \boldsymbol{w}_2^T Y^T S_2^{-1} Y\boldsymbol{w}_2 \\ &\quad + \cdots + \boldsymbol{w}_m^T Y^T S_2^{-1} Y\boldsymbol{w}_m \\ &= \begin{bmatrix}\boldsymbol{w}_1^T Y^T \ \boldsymbol{w}_2^T Y^T \cdots \boldsymbol{w}_m^T Y^T\end{bmatrix} S_2^{-1} \begin{bmatrix} Y\boldsymbol{w}_1 \\ Y\boldsymbol{w}_2 \\ \vdots \\ Y\boldsymbol{w}_m\end{bmatrix} < \gamma_4\end{aligned} \tag{55}$$

Further, Schur complement derives (15) from (55).

□

It is noted that the inequalities (11)-(13) cannot be solved directly by LMI because they contain the matrices S_1, X, X_l, $(X_l)_{inv}$, Y_l, $(Y_l)_{inv}$ and scalars such as δ_7, δ_{7inv}, δ_8, δ_{8inv} and so on which satisfy the inverse relations. In addition, nonlinear terms also appeared. Therefore, algorithm in [8] should be modified. Here, we apply the cone complementarity linearization approach and propose the iterative algorithm to solve.

4. A NUMERICAL EXAMPLE

Consider a continuous-time system with the following transfer function

$$G(s) = \frac{1+\beta}{s(s+1+\alpha)}$$

where α and β are the uncertainties.

We can convert it to the state space form as follows.

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \begin{bmatrix}-1-\alpha & 0 \\ 1 & 0\end{bmatrix}\boldsymbol{x}(t) + \begin{bmatrix}1+\beta \\ 0\end{bmatrix}\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \begin{bmatrix}0 & 1\end{bmatrix}\boldsymbol{x}(t)\end{aligned}$$

Then, the discrete-time form is

$$\begin{aligned}\boldsymbol{x}(k+1) &= \left\{\begin{bmatrix}1-T_s & 0 \\ T_s & 1\end{bmatrix} + \begin{bmatrix}\Delta a_1 & 0 \\ \Delta a_2 & 0\end{bmatrix}\right\}\boldsymbol{x}(k) \\ &\quad + \left\{\begin{bmatrix}T_s \\ 0\end{bmatrix} + \begin{bmatrix}\Delta b_1 \\ \Delta b_2\end{bmatrix}\right\}\boldsymbol{u}(k) \\ \boldsymbol{y}(k) &= \begin{bmatrix}0 & 1\end{bmatrix}\boldsymbol{x}(k)\end{aligned}$$

with $T_s = 0.1$, $\Delta a_1 = 0.006$, $\Delta a_2 = -0.006$, $\Delta b_1 = \Delta b_2 = 0$,

The following parameters are given

$$\boldsymbol{m}_0 = \begin{bmatrix}0 \\ 0\end{bmatrix},\ \Sigma_0 = \begin{bmatrix}0.1 & 0 \\ 0 & 0.1\end{bmatrix},\ R = 0.01,\ Q = \begin{bmatrix}0.1 & 0 \\ 0 & 0.2\end{bmatrix}.$$

Applying Theorem 1 with j_{max}=1000, γ_{min}=0.0001, κ=0.0001 and initial γ=2, there exist symmetric positive definite matrices and scalars

$$S_1 = \begin{bmatrix}0.59437490 & 0.06831873 \\ 0.06831873 & 0.59066350\end{bmatrix},$$

$$X = \begin{bmatrix}1.70510875 & -0.19722036 \\ -0.19722036 & 1.71582268\end{bmatrix},$$

$$X_1 = \begin{bmatrix}1.01021505 & -0.01003115 \\ -0.01003115 & 1.00987028\end{bmatrix},$$

$$X_{1inv} = \begin{bmatrix}0.98998587 & 0.00983363 \\ 0.00983363 & 0.99032386\end{bmatrix},$$

$$X_2 = \begin{bmatrix}1.01266942 & -0.01244290 \\ -0.01244290 & 1.01224011\end{bmatrix},$$

$$X_{2inv} = \begin{bmatrix}0.98763825 & 0.01214048 \\ 0.01214048 & 0.98805712\end{bmatrix},$$

$$X_3 = \begin{bmatrix}1.01266945 & -0.01244292 \\ -0.01244292 & 1.01224015\end{bmatrix},$$

$$X_{3inv} = \begin{bmatrix}0.98763822 & 0.01214050 \\ 0.01214050 & 0.98805708\end{bmatrix},$$

$$X_4 = \begin{bmatrix}1.01274069 & -0.01251263 \\ -0.01251263 & 1.01230841\end{bmatrix},$$

$$X_{4inv} = \begin{bmatrix}0.98757041 & 0.01220686 \\ 0.01220686 & 0.98799212\end{bmatrix},$$

$$X_5 = \begin{bmatrix}1.01274065 & -0.01251263 \\ -0.01251263 & 1.01230838\end{bmatrix},$$

$$X_{5inv} = \begin{bmatrix}0.98757044 & 0.01220686 \\ 0.01220686 & 0.98799215\end{bmatrix},$$

$$Y_1 = \begin{bmatrix}1.01020000 & -0.01001977 \\ -0.01001977 & 1.00984678\end{bmatrix},$$

$$Y_{1inv} = \begin{bmatrix}0.99000041 & 0.00982286 \\ 0.00982286 & 0.99034668\end{bmatrix},$$

$$Y_2 = \begin{bmatrix}1.01265435 & -0.01243153 \\ -0.01243153 & 1.01221660\end{bmatrix},$$

$$Y_{2inv} = \begin{bmatrix}0.98765268 & 0.01212985 \\ 0.01212985 & 0.98807980\end{bmatrix},$$

$$Y_3 = \begin{bmatrix}1.01265439 & -0.01243155 \\ -0.01243155 & 1.01221662\end{bmatrix},$$

$$Y_{3inv} = \begin{bmatrix}0.98765264 & 0.01212987 \\ 0.01212987 & 0.98807978\end{bmatrix},$$

$$Y_4 = \begin{bmatrix}1.01272557 & -0.01250124 \\ -0.01250124 & 1.01228488\end{bmatrix},$$

$$Y_{4inv} = \begin{bmatrix}0.98758487 & 0.01219620 \\ 0.01219620 & 0.98801482\end{bmatrix},$$

$$Y_5 = \begin{bmatrix}1.01272556 & -0.01250124 \\ -0.01250124 & 1.01228487\end{bmatrix},$$

$$Y_{5inv} = \begin{bmatrix}0.98758488 & 0.01219620 \\ 0.01219620 & 0.98801482\end{bmatrix},$$

$$\begin{aligned}\delta_7 &= 0.99999956,\ \delta_{7inv} = 1.00000043, \\ \delta_8 &= 0.99999954,\ \delta_{8inv} = 1.00000045, \\ \delta_{10} &= 1.01620314,\ \delta_{10inv} = 0.98405521, \\ \delta_{11} &= 1.01881851,\ \delta_{11inv} = 0.98152908, \\ \delta_{19} &= 1.01881845,\ \delta_{19inv} = 0.98152913, \\ \delta_{20} &= 1.00000016,\ \delta_{20inv} = 0.99999983, \\ \delta_{26} &= 1.22843365,\ \delta_{26inv} = 0.81404477, \\ \delta_{27} &= 0.98405535,\ \delta_{27inv} = 1.01620299,\end{aligned}$$

$\delta_{29} = 0.99551922,\ \delta_{29inv} = 1.00450094,$
$\delta_{31} = 0.98152928,\ \delta_{31inv} = 1.01881829,$

which satisfy LMIs in (11)-(15) and their inverse relations.

Further, we obtain a solution

$$L = 0.25348223,$$
$$K = \begin{bmatrix} -0.04861597 & 0.19639255 \\ -0.06175519 & -0.39112435 \end{bmatrix},$$
$$E = 0.51274725,$$
$$H_1 = 0.10000000,\ H_0 = 0.09253482,$$
$$W = \begin{bmatrix} -0.25348223 \\ 1.00000000 \end{bmatrix},\ \bar{\gamma} = E[J^*] = 0.140625.$$

Figure 1 displays the transisition of guaranteed cost. The mark + shows that a feasible solution cannot be obtained for the guaranteed cost candidate $\bar{\gamma}$ and the optimal performance index is greater than +. Trajectories of states and estimate error are depicted in Figs. 2-3 with $x(0) = [-0.1\ \ 0.2]^T$. The control inputs are illustrated in Fig. 4

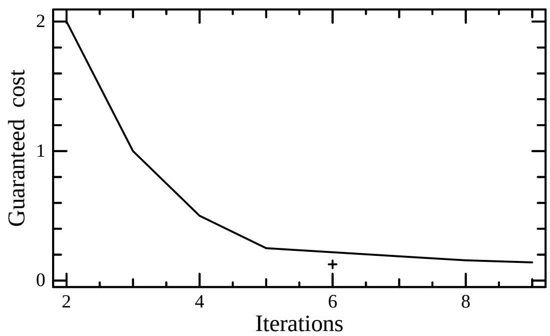

Fig. 1 Trajectory of the guaranteed cost.

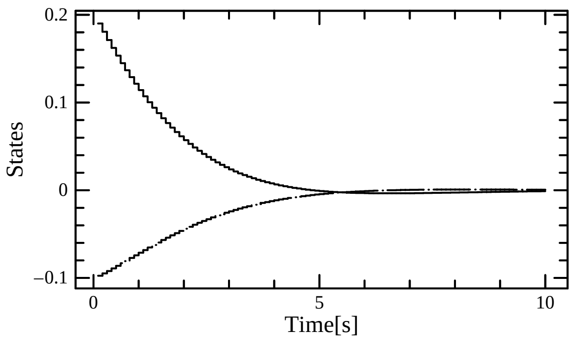

Fig. 2 Trajectories of states x_1 (-·-) and x_2 (–).

5. CONCLUSION

This paper discusses a minimal order observer-based guaranteed cost control design for multirate systems. A sufficient condition for the existence of state feedback guaranteed cost controllers is derived on the basis of the LMI feasible solutions.

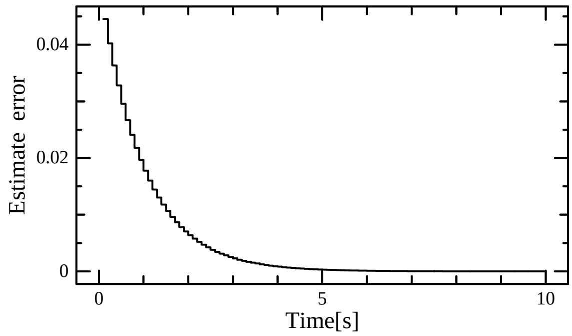

Fig. 3 Trajectory of estimate error ξ.

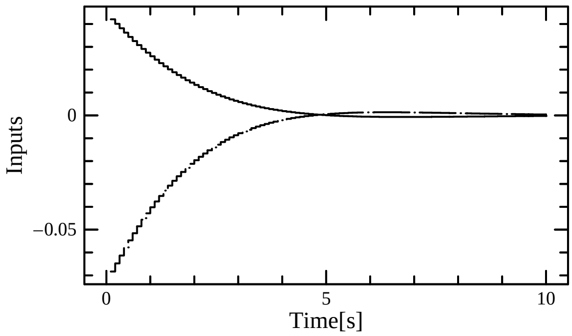

Fig. 4 Control inputs $u(k)$ (-·-) and $u(k+1)$ (–).

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