SICE Annual Conference 2013
September 14-17, 2013, Nagoya University, Nagoya, Japan

An LMI approach to optimal guaranteed cost control of multirate sampling systems

Mitsuaki Ishitobi[1], Tatsuhiro Tamura[1], Erwin Susanto[2] and Sadaaki Kunimatsu[1]

[1]Department of Mechanical Systems Engineering, Kumamoto University, Kumamoto, Japan
(Tel: +81-96-342-3777; E-mail: mishi@kumamoto-u.ac.jp)
[2]Department of Electrical Engineering, Telkom Institute of Technology, Bandung, Indonesia

Abstract: This paper considers a design method of a guaranteed cost controller for multirate sampling systems. A minimal order multirate observer is used to estimate a state vector at the output sampling period. The controller and the observer are obtained by using a linear matrix inequality technique. A numerical example illustrates the effectiveness of the proposed method.

Keywords: guaranteed cost control, multirate systems, LMI, a minimal order observer

1. INTRODUCTION

Instability and bad performance can occur in a closed-loop feedback control system with uncertainties. Therefore, considerable interests have been attracted to studies of robust controller design in recent decades. Moreover, it is desirable to design a controller which not only achieves the stability of the uncertain system but also guarantees an adequate level of performance. One of the approaches to solve this problem is a guaranteed cost control method [1]. Many significant results have been shown for the continuous-time case [2], [3] and for the single rate sampling discrete-time case [4], [5], [6].

Multirate sampling schemes have long been the focus of interest by many control designers [7], [8], [9], [10]. This paper considers a design method of a guaranteed cost controller for multirate systems. The controller is obtained in the basis of a multirate state-space model and the state variable is estimated by a minimal order multirate observer. The design problem is expressed by matrix inequalities and solved by an algorithm of a linear matrix inequality technique. This paper extends the previous results [11] to systems with uncertainties in the input term.

2. PROBLEM STATEMENT

Consider a discrete-time uncertain system in the form

$$\begin{aligned} \boldsymbol{x}(k+1) &= (A_0+\Delta A_0)\boldsymbol{x}(k)+(B_0+\Delta B_0)\boldsymbol{u}(k) \\ \boldsymbol{y}(k) &= C\boldsymbol{x}(k) \end{aligned} \tag{1}$$

Assume the output sampling period is greater than that of the input and only $\boldsymbol{y}(iN)$ is available for $i = 0,\ 1,\ \cdots$ where N is a positive integer greater than one and iN means the time iNT_s where T_s denotes a sampling period. Then, the output-based form of (1) can be written as follows

$$\begin{aligned} \boldsymbol{x}(k+N) &= (A+\Delta A)\boldsymbol{x}(k)+(B+\Delta B)\boldsymbol{v}(k) \\ \boldsymbol{y}(k) &= C\boldsymbol{x}(k) \end{aligned} \tag{2}$$

where

$$\begin{aligned} A &= A_0^N \\ \Delta A &= (A_0+\Delta A_0)^N - A_0^N \end{aligned} \tag{3}$$

$$\begin{aligned} B &= [B_0, A_0B_0, \cdots, A_0^{N-1}B_0] \\ \Delta B &= [\Delta B_0, \\ &\quad (A_0+\Delta A_0)(B_0+\Delta B_0)-A_0B_0, \\ &\quad \cdots, (A_0+\Delta A_0)^{N-1}(B_0+\Delta B_0) \\ &\quad -A_0^{N-1}B_0] \\ \boldsymbol{v}(k) &= [\boldsymbol{u}^T(k+N-1), \boldsymbol{u}^T(k+N-2), \\ &\quad \cdots, \boldsymbol{u}^T(k)]^T \end{aligned} \tag{4}$$

Matrices A_0, and B_0 are known constant real-valued matrices with appropriate dimensions, and C is restricted to the form of $C = [O\ \ I_m]$.

We assume that the parameter uncertainties ΔA_0 and ΔB_0 satisfy the following relations

$$\Delta A_0 = D_A F_A E_A,\ \Delta B_0 = D_B F_B E_B \tag{5}$$

where F_A and F_B are unknown and deterministic matrices satisfying

$$F_A^T F_A \leq I,\ F_B^T F_B \leq I \tag{6}$$

and D_A, E_A are constant real-valued known matrices with appropriate dimensions.

It is also assumed that the initial state variable $\boldsymbol{x}(0)$ is unknown, but their mean and covariance are known, respectively as

$$E\left[\boldsymbol{x}(0)\right] = \boldsymbol{m}_0 \tag{7}$$

$$E\left[(\boldsymbol{x}(0)-\boldsymbol{m}_0)(\boldsymbol{x}(0)-\boldsymbol{m}_0)^T\right] = \Sigma_0 > O \tag{8}$$

where $E\left[\cdot\right]$ denotes the expected value operator.

The problem considered here is to design a minimal order observer

$$\boldsymbol{z}(k+N) = D\boldsymbol{z}(k)+E\boldsymbol{y}(k)+\sum_{i=0}^{N-1} H_i\boldsymbol{u}(k+i) \tag{9}$$

$$\hat{\boldsymbol{x}}(k) = P\boldsymbol{z}(k)+W\boldsymbol{y}(k) \tag{10}$$

and a controller

$$\boldsymbol{v}(k) = K\hat{\boldsymbol{x}}(k) \tag{11}$$

with

$$K=\begin{bmatrix}K_{N-1}\\ \vdots\\ K_1\\ K_0\end{bmatrix},\ A=\begin{bmatrix}A_{11}\ A_{12}\\ A_{21}\ A_{22}\end{bmatrix},\ D=A_{11}+LA_{21},$$
$$H_i=TA_0^{N-1-i}B_0,\ TA-DT=EC,$$
$$T=\begin{bmatrix}I_{n-m}\ L\end{bmatrix},\ PT+WC=I_n,\ P=\begin{bmatrix}I_{n-m}\ O\end{bmatrix}^T$$

so as to achieve an upper bound on the following quadratic performance index

$$E[J]=E\left[\sum_{l=0}\{\boldsymbol{x}^T(Nl)Q\boldsymbol{x}(Nl)+\boldsymbol{v}^T(Nl)R\boldsymbol{v}(Nl)\}\right] \tag{12}$$

associated with the multirate systems (2), where Q and R are given symmetric positive-definite matrices.

3. MAIN RESULTS

Attention of this study is restricted to $N=2$ for simplicity of description. Extension to the general case of $N\geq 3$ is complicated though the basic idea is the same. The main result of this study is given by Theorem 1.

Theorem 1. If the following matrix inequalities optimization problem; $\min\{\gamma_0+\gamma_1\}$ subject to

$$\begin{bmatrix}\Lambda_{11} & 0 & A^T+K^TB^T & 0 & K^T\\ & \Lambda_{22} & P^TK^TB^T & A_{11}^T+A_{21}^TL^T & P^TK^T\\ & & \Lambda_{33} & 0 & 0\\ & & & \Lambda_{44} & 0\\ * & & & & -R^{-1}\end{bmatrix}<0 \tag{13}$$

$$\begin{aligned}\Lambda_{11}=&-S_1+Q+Y_1+Y_2+Y_3+Y_4+Y_5\\&+Y_{10}+Y_{12}+Y_{14}+Y_{16}\\&+(\epsilon_{1,inv}+\epsilon_{4,inv}+\epsilon_{6,inv})E_A^TE_A\\&+(\epsilon_{2,inv}+\epsilon_{5,inv})A_0^TE_A^TE_AA_0\\ \Lambda_{22}=&-S_2+Y_{17}+Y_{18}+Y_{19}+Y_{20}+Y_{22}\\&+Y_{24}+Y_{26}+Y_{28}\\ \Lambda_{33}=&-S_{1,inv}+(\epsilon_2+\epsilon_3+\delta_3+\delta_4+\delta_{11}\\&+\delta_{12})D_AD_A^T+\epsilon_1A_0D_AD_A^TA_0^T\\&+(\delta_1+\delta_9)D_BD_B^T\\&+(\delta_2+\delta_{10})A_0D_BD_B^TA_0^T\\ \Lambda_{44}=&-S_{2,inv}+Y_6+Y_7+Y_8+Y_9+Y_{11}\\&+Y_{13}+Y_{15}+Y_{21}+Y_{23}+Y_{25}+Y_{27}\end{aligned}$$

$$\begin{bmatrix}-Y_1+\mu_{6,inv}E_A^TE_A & 0\\ 0 & -\epsilon_3I+\mu_6E_AD_AD_A^TE_A^T\end{bmatrix}\leq 0$$
$$\begin{bmatrix}-Y_5+X_1 & 0\\ 0 & -\delta_4I+\mu_1E_AD_BE_A^TD_B^T\end{bmatrix}\leq 0$$
$$\begin{bmatrix}-Y_8+X_2 & 0\\ 0 & -\epsilon_{6,inv}I+\mu_2D_A^TE_A^TE_AD_A\end{bmatrix}\leq 0$$
$$\begin{bmatrix}-Y_{16}+X_3 & 0\\ 0 & -\delta_8I+\mu_3E_AD_BE_A^TD_B^T\end{bmatrix}\leq 0$$
$$\begin{bmatrix}-Y_{20}+X_4 & 0\\ 0 & -\delta_{12}I+\mu_4E_AD_BE_A^TD_B^T\end{bmatrix}\leq 0$$
$$\begin{bmatrix}-Y_{28}+X_5 & 0\\ 0 & -\delta_{16}I+\mu_5E_AD_BE_A^TD_B^T\end{bmatrix}\leq 0$$
$$\begin{bmatrix}-Y_2 & K_1^TE_B^T\\ * & -\delta_1I\end{bmatrix}\leq 0,\quad \begin{bmatrix}-Y_3 & K_2^TE_B^T\\ * & -\delta_2I\end{bmatrix}\leq 0$$
$$\begin{bmatrix}-Y_4 & K_2^TB_0^TE_B^T\\ * & -\delta_3I\end{bmatrix}\leq 0,\quad \begin{bmatrix}-Y_6 & TA_0D_A\\ * & -\epsilon_{4,inv}I\end{bmatrix}\leq 0$$
$$\begin{bmatrix}-Y_7 & TD_A\\ * & -\epsilon_{5,inv}I\end{bmatrix}\leq 0,\quad \begin{bmatrix}-Y_9 & TD_B\\ * & -\delta_{5,inv}I\end{bmatrix}\leq 0$$
$$\begin{bmatrix}-Y_{10} & K_1^TE_B^T\\ * & -\delta_5I\end{bmatrix}\leq 0,\quad \begin{bmatrix}-Y_{11} & TA_0D_B\\ * & -\epsilon_{6,inv}I\end{bmatrix}\leq 0$$
$$\begin{bmatrix}-Y_{12} & K_2^TE_B^T\\ * & -\delta_6I\end{bmatrix}\leq 0,\quad \begin{bmatrix}-Y_{13} & TD_A\\ * & -\delta_{7,inv}I\end{bmatrix}\leq 0$$
$$\begin{bmatrix}-Y_{14} & K_2^TB_0^TE_A^T\\ * & -\delta_7I\end{bmatrix}\leq 0,\quad \begin{bmatrix}-Y_{15} & TD_A\\ * & -\delta_{8,inv}I\end{bmatrix}\leq 0$$
$$\begin{bmatrix}-Y_{17} & P^TE_B^T\\ * & -\delta_9I\end{bmatrix}\leq 0,\quad \begin{bmatrix}-Y_{18} & P^TK_2^TE_B^T\\ * & -\delta_{10}I\end{bmatrix}\leq 0$$
$$\begin{bmatrix}-Y_{19} & P^TK_2^TB_0^TE_A^T\\ * & -\delta_{11}I\end{bmatrix}\leq 0,\quad \begin{bmatrix}-Y_{21} & TD_B\\ * & -\delta_{13,inv}I\end{bmatrix}\leq 0$$
$$\begin{bmatrix}-Y_{22} & P^TK_1^TE_B^T\\ * & -\delta_{13}I\end{bmatrix}\leq 0,\quad \begin{bmatrix}-Y_{23} & TA_0D_B\\ * & -\delta_{14,inv}I\end{bmatrix}\leq 0$$
$$\begin{bmatrix}-Y_{24} & P^TK_2^TE_B^T\\ * & -\delta_{14}I\end{bmatrix}\leq 0,\quad \begin{bmatrix}-Y_{25} & TD_A\\ * & -\delta_{15,inv}I\end{bmatrix}\leq 0$$
$$\begin{bmatrix}-Y_{26} & P^TK_2^TB_0^TE_A^T\\ * & -\delta_{15}I\end{bmatrix}\leq 0,\quad \begin{bmatrix}-Y_{27} & TD_A\\ * & -\delta_{16,inv}I\end{bmatrix}\leq 0 \tag{14}$$

$$\begin{bmatrix}-X_1 & K_2^TE_B^T\\ * & -\mu_1I\end{bmatrix}\leq 0,\quad \begin{bmatrix}-X_2 & TD_A\\ * & -\mu_2I\end{bmatrix}\leq 0$$
$$\begin{bmatrix}-X_3 & K_2^TE_B^T\\ * & -\mu_3I\end{bmatrix}\leq 0,\quad \begin{bmatrix}-X_4 & P^TK_2^TE_B^T\\ * & -\mu_4I\end{bmatrix}\leq 0$$
$$\begin{bmatrix}-X_5 & P^TK_2^TE_B^T\\ * & -\mu_5I\end{bmatrix}\leq 0 \tag{15}$$

$$-\gamma_0+\boldsymbol{v}_1^TS_1\boldsymbol{v}_1+\boldsymbol{v}_2^TS_1\boldsymbol{v}_2+\cdots++\boldsymbol{v}_n^TS_1\boldsymbol{v}_n<0 \tag{16}$$

$$\begin{bmatrix}-\gamma_1 & \boldsymbol{w}_2^TT^T & \boldsymbol{w}_2^TT^T & \cdots & \boldsymbol{w}_n^TT^T\\ * & -S_{2,inv} & & & \\ * & & -S_{2,inv} & & \\ \vdots & & & \ddots & \\ * & & & & -S_{2,inv}\end{bmatrix}<0 \tag{17}$$

where

$$[\boldsymbol{v}_1\ \boldsymbol{v}_2\ \cdots\ \boldsymbol{v}_n]:=(\Sigma_0+\boldsymbol{m}_0\boldsymbol{m}_0^T)^{\frac{1}{2}} \tag{18}$$
$$[\boldsymbol{w}_1\ \boldsymbol{w}_2\ \cdots\ \boldsymbol{w}_n]:=\Sigma_0^{\frac{1}{2}} \tag{19}$$

has a set of solutions $S_1>0$, $S_2>0$, $S_{1,inv}>0$, $S_{2,inv}>0$, K, L, T, $X_1,\cdots,X_5$, $Y_1,\cdots,Y_{28}$, $\epsilon_1>0$,

$\cdots$, $\epsilon_3 > 0$, $\epsilon_{1,inv} > 0$, $\epsilon_{2,inv} > 0$, $\epsilon_{4,inv} > 0$, $\cdots$, $\epsilon_{6,inv} > 0$, $\delta_1 > 0$, $\cdots$, $\delta_{16} > 0$, $\delta_{5,inv} > 0$, $\cdots$, $\delta_{8,inv} > 0$, $\delta_{13,inv} > 0$, $\cdots$, $\delta_{16,inv} > 0$, $\mu_1 > 0$, $\cdots$, $\mu_5 > 0$, γ_0, γ_1, which satisfy the inverse relation such as $S_1^{-1} = S_{1,inv}$, $S_2^{-1} = S_{2,inv}$, $\epsilon_1^{-1} = \epsilon_{1,inv}$, $\epsilon_2^{-1} = \epsilon_{2,inv}$, $\delta_5^{-1} = \delta_{5,inv}$, $\cdots$, $\delta_8^{-1} = \delta_{8,inv}$, $\delta_{13}^{-1} = \delta_{13,inv}$, $\cdots$, $\delta_{16}^{-1} = \delta_{16,inv}$, $\mu_6^{-1} = \mu_{6,inv}$, then the minimal order observer-based control law (9)-(11) is a guaranteed cost controller which gives the minimum expected value of the guaranteed cost

$$E[J^*] = E\left[\boldsymbol{x}^T(0)Q\boldsymbol{x}(0) + \boldsymbol{v}^T(0)R\boldsymbol{v}(0))\right] \quad (20)$$

Remark 1: Since (13)-(17) have a constraint of the inverse relations and nonlinear terms, a cone complementarity linealization algorithm is introduced to solve [12], [13].

Before giving a proof of Theorem 1, a key lemma is introduced.

Lemma 1 [14]. Given matrices D and E of appropriate dimensions, and F be a matrix function satisfying $F^TF \le I$, then for any $\alpha > 0$, the following inequality holds

$$DFE + E^TF^TD^T \le \alpha DD^T + \alpha^{-1}E^TE.$$

Proof of Theorem 1.

From (11) we have

$$\begin{aligned}\boldsymbol{v}(k) =& K\hat{\boldsymbol{x}}(k)\\ =& K\{P\boldsymbol{\xi}(k) + \boldsymbol{x}(k)\}\end{aligned} \quad (21)$$

where $\boldsymbol{\xi}(k) = \boldsymbol{z}(k) - T\boldsymbol{x}(k)$ is the estimated error of the minimal order observer. Then, using (21) and (2), we obtain

$$\begin{aligned}\boldsymbol{x}(k+2) =& (A+\Delta A)\boldsymbol{x}(k) + (B+\Delta B)\boldsymbol{v}(k)\\ =& \{(A+\Delta A) + (B+\Delta B)K\}\boldsymbol{x}(k)\\ &+ (B+\Delta B)KP\boldsymbol{\xi}(k)\\ \boldsymbol{\xi}(k+2) =& \boldsymbol{z}(k+2) - T\boldsymbol{x}(k+2)\\ =& \{-T\Delta A - T\Delta BK\}\boldsymbol{x}(k)\\ &+ (D - T\Delta BKP)\boldsymbol{\xi}(k)\end{aligned}$$

Thus, the closed-loop system is expressed as

$$\boldsymbol{w}(k+2) = G\boldsymbol{w}(k) \quad (22)$$

where

$$\boldsymbol{w}(k) := \begin{bmatrix}\boldsymbol{x}(k)\\ \boldsymbol{\xi}(k)\end{bmatrix} \quad (23)$$

$$G := \begin{bmatrix}G_{11} & G_{12}\\ G_{21} & G_{22}\end{bmatrix} \quad (24)$$

where

$$\begin{aligned}G_{11} &:= (A+\Delta A) + (B+\Delta B)K\\ G_{12} &:= (B+\Delta B)KP\\ G_{21} &:= -T\Delta A - T\Delta BK\\ G_{22} &:= D - T\Delta BKP\end{aligned}$$

Define a candidate of a Lyapunov function as

$$\begin{aligned}V(k) &= \boldsymbol{x}^T(k)S_1\boldsymbol{x}(k) + \boldsymbol{\xi}^T(k)S_2\boldsymbol{\xi}(k)\\ &= \boldsymbol{w}^T(k)S\boldsymbol{w}(k)\end{aligned} \quad (25)$$

where

$$S = \text{block-diag}(S_1, S_2) \quad (26)$$

and $S_1 = S_1^T > 0$, $S_2 = S_2^T > 0$, then the forward difference of $V(k)$ is obtained as

$$\begin{aligned}\Delta V(k) =& V(k+2) - V(k)\\ =& \boldsymbol{w}^T(k)(G^TSG - S)\boldsymbol{w}(k)\end{aligned} \quad (27)$$

In addition, since it holds that

$$\boldsymbol{x}^T(k)Q\boldsymbol{x}(k) + \boldsymbol{v}^T(k)R\boldsymbol{v}(k) = \boldsymbol{w}^T(k)\Phi\boldsymbol{w}(k) \quad (28)$$

where

$$\Phi = \begin{bmatrix}Q + K^TRK & K^TRKP\\ * & P^TK^TRKP\end{bmatrix} \quad (29)$$

(25) and (27) lead to

$$\begin{aligned}\Delta V(k) =& \boldsymbol{w}^T(k)(G^TSG - S)\boldsymbol{w}(k)\\ =& \boldsymbol{w}^T(k)\Omega\boldsymbol{w}(k)\\ &- (\boldsymbol{x}^T(k)Q\boldsymbol{x}(k) + \boldsymbol{v}^T(k)R\boldsymbol{v}(k))\end{aligned} \quad (30)$$

where

$$\Omega = G^TSG - S + \Phi \quad (31)$$

If Ω satisfies

$$\Omega < 0 \quad (32)$$

(30) yields

$$\Delta V(k) < -(\boldsymbol{x}^T(k)Q\boldsymbol{x}(k) + \boldsymbol{v}^T(k)R\boldsymbol{v}(k)) < 0 \qquad {}^\forall\boldsymbol{w}(k) \neq \boldsymbol{0} \quad (33)$$

and the closed-loop system is asymptotically stable.

We can decompose (29) as

$$\Phi = \begin{bmatrix}Q & 0\\ 0 & 0\end{bmatrix} + \begin{bmatrix}K^T\\ P^TK^T\end{bmatrix} R \begin{bmatrix}K & KP\end{bmatrix} \quad (34)$$

Thus, substituting (31) and (34) into (32) we know that the stability condition for this problem is expressed as

$$\begin{aligned}\Omega =\ & G^TSG - S\\ &+ \begin{bmatrix}Q & 0\\ 0 & 0\end{bmatrix} + \begin{bmatrix}K^T\\ P^TK^T\end{bmatrix} R \begin{bmatrix}K & KP\end{bmatrix} < 0\end{aligned} \quad (35)$$

Then substituting (24), (26), (3) and (4) into (35), and applying Schur Complement, we obtain the following condition.

$$M := \begin{bmatrix}-S_1 + Q & 0 & M_{13} & M_{14} & K^T\\ & -S_2 & M_{23} & M_{24} & P^TK^T\\ & & -S_1^{-1} & 0 & 0\\ & & & -S_2^{-1} & 0\\ * & & & & -R^{-1}\end{bmatrix} < 0 \quad (36)$$

where

$$\begin{aligned}M_{13} =& A^T + K^T B^T + \Delta A_0^T A_0^T + A_0^T \Delta A_0^T \\ &+ (\Delta A_0^T)^2 + K_1^T \Delta B_0^T + K_2^T \Delta B_0^T A_0^T \\ &+ K_2^T B_0^T \Delta A_0^T + K_2^T \Delta B_0^T \Delta A_0^T \\ M_{14} =& -\Delta A_0^T A_0^T T^T - A_0^T \Delta A_0^T T^T \\ &- (\Delta A_0^T)^2 T^T - K_1^T \Delta B_0^T T^T \\ &- K_2^T \Delta B_0^T A_0^T T^T - K_2^T B_0^T \Delta A_0^T T^T \\ &- K_2^T \Delta B_0^T \Delta A_0^T T^T \\ M_{23} =& P^T K^T B^T + P^T K_1^T \Delta B_0^T \\ &+ P^T K_2^T \Delta B_0^T A_0^T + P^T K_2^T B_0^T \Delta A_0^T \\ &+ P^T K_2^T \Delta B_0^T \Delta A_0^T \\ M_{24} =& D^T - P^T K_1^T \Delta B_0^T T^T \\ &- P^T K_2^T \Delta B_0^T A_0^T T^T \\ &- P^T K_2^T B_0^T \Delta A_0^T T^T \\ &- P^T K_2^T \Delta B_0^T \Delta A_0^T T^T\end{aligned}$$

Moreover□consider a quadratic form $\bar{\boldsymbol{x}}^T M \bar{\boldsymbol{x}}$ where $\bar{\boldsymbol{x}} = [\boldsymbol{x}_1^T, \boldsymbol{x}_2^T, \boldsymbol{x}_3^T, \boldsymbol{x}_4^T, \boldsymbol{x}_5^T]^T$, and $\boldsymbol{x_1} \in R^n$, $\boldsymbol{x_2} \in R^{n-m}$, $\boldsymbol{x_3} \in R^n$, $\boldsymbol{x_4} \in R^n$ and $\boldsymbol{x_5} \in R^r$ are arbitrary nonzero vectors.

Then applying Lemma 1 to the above quadratic form, it holds for any $\epsilon_1 > 0$ that

$$\begin{aligned}&2\boldsymbol{x_1}^T \Delta A_0^T A_0^T \boldsymbol{x_3} \\ &\le \epsilon_1 \boldsymbol{x_3}^T A_0 D_A D_A^T A_0^T \boldsymbol{x_3} + \epsilon_1^{-1} \boldsymbol{x_1}^T E_A^T E_A \boldsymbol{x_1}\end{aligned} \tag{37a}$$

Similar inequality results are obtained for $\epsilon_2 > 0, \cdots, \epsilon_6 > 0, \delta_1 > 0, \cdots, \delta_{16} > 0$. Then, a sufficient matrix inequality condition of (36) is derived. Moreover, applying Schur Complement and Lemma 1 leads to (13), (14) and (15). Refer to [15] for details of derivation of (13), (14) and (15).

Further□ summing (33) from 0 to $\bar{N}$ yields

$$\begin{aligned}&\boldsymbol{x}^T(\bar{N}+1) S_1 \boldsymbol{x}(\bar{N}+1) + \boldsymbol{\xi}^T(\bar{N}+1) S_2 \boldsymbol{\xi}(\bar{N}+1) \\ &-(\boldsymbol{x}^T(0) S_1 \boldsymbol{x}(0) + \boldsymbol{\xi}^T(0) S_2 \boldsymbol{\xi}(0)) \\ &< -\sum_{l=0}^{\bar{N}} (\boldsymbol{x}^T(Nl) Q \boldsymbol{x}(Nl) + \boldsymbol{v}^T(Nl) R \boldsymbol{v}(Nl)) < 0\end{aligned} \tag{38}$$

Here, the asymptotic stability of the closed-loop system implies that

$$\boldsymbol{x}^T(\bar{N}+1) S_1 \boldsymbol{x}(\bar{N}+1) \to 0,\ \boldsymbol{\xi}^T(\bar{N}+1) S_1 \boldsymbol{\xi}(\bar{N}+1) \to 0 \tag{39}$$

as $\bar{N}$ tends to the infinity. Hence, it is obtained that

$$\begin{aligned}J &= \sum_{l=0}^{\infty} (\boldsymbol{x}^T(Nl) Q \boldsymbol{x}(Nl) + \boldsymbol{v}^T(Nl) R \boldsymbol{v}(Nl)) \\ &< \boldsymbol{x}^T(0) S_1 \boldsymbol{x}(0) + \boldsymbol{\xi}^T(0) S_2 \boldsymbol{\xi}(0) \\ &= J^*\end{aligned} \tag{40}$$

where J^* denotes the guaranteed cost. Here, we consider the optimal expected value of the guaranteed cost. It is calculated as

$$\begin{aligned}E[J^*] &= E\left[\boldsymbol{x}^T(0) S_1 \boldsymbol{x}(0) + \boldsymbol{\xi}^T(0) S_2 \boldsymbol{\xi}(0)\right] \\ &= \mathrm{tr} S_1 E\left[\boldsymbol{x}(0)\boldsymbol{x}^T(0)\right] + \mathrm{tr} S_2 E\left[\boldsymbol{\xi}(0)\boldsymbol{\xi}^T(0)\right]\end{aligned} \tag{41}$$

A relation between mean and covarience of $\boldsymbol{x}(0)$ is given by

$$\Sigma_0 = E\left[\boldsymbol{x}(0)\boldsymbol{x}^T(0)\right] - \boldsymbol{m}_0 \boldsymbol{m}_0^T \tag{42}$$

Substituting (42) into (41) yields

$$\begin{aligned}E[J^*] =& \mathrm{tr} S_1(\Sigma_0 + \boldsymbol{m}_0 \boldsymbol{m}_0^T) \\ &+ \mathrm{tr} S_2 E\left[(\boldsymbol{z}(0) - T\boldsymbol{x}(0))(\boldsymbol{z}(0) - T\boldsymbol{x}(0))^T\right]\end{aligned} \tag{43}$$

Here, it is readily seen that

$$\begin{aligned}&E\left[(\boldsymbol{z}(0) - T\boldsymbol{x}(0))(\boldsymbol{z}(0) - T\boldsymbol{x}(0))^T\right] \\ &= T\Sigma_0 T^T + (\boldsymbol{z}(0) - T\boldsymbol{m}_0)(\boldsymbol{z}(0) - T\boldsymbol{m}_0)^T\end{aligned} \tag{44}$$

Hence□ (43) leads to

$$\begin{aligned}E[J^*] =& \mathrm{tr} S_1(\Sigma_0 + \boldsymbol{m}_0 \boldsymbol{m}_0^T) + \mathrm{tr} S_2(T\Sigma_0 T^T \\ &+ (\boldsymbol{z}(0) - T\boldsymbol{m}_0)(\boldsymbol{z}(0) - T\boldsymbol{m}_0)^T)\end{aligned} \tag{45}$$

Here, it can be assumed that an initial value $\boldsymbol{z}(0)$ of the minimal order observer satisfies the following equation without loss of generality.

$$\boldsymbol{z}(0) - T\boldsymbol{m}_0 = \boldsymbol{0} \tag{46}$$

Substituting (46) into (45) yields

$$E[J^*] = \mathrm{tr} S_1(\Sigma_0 + \boldsymbol{m}_0 \boldsymbol{m}_0^T) + \mathrm{tr} S_2 T\Sigma_0 T^T \tag{47}$$

Here, we consider positive scalars γ_0, γ_1 satisfying the following inequalities.

$$\mathrm{tr} S_1(\Sigma_0 + \boldsymbol{m}_0 \boldsymbol{m}_0^T) < \gamma_0 \tag{48}$$

$$\mathrm{tr} S_2 T\Sigma_0 T^T < \gamma_1 \tag{49}$$

Then, minimizing $\gamma_0 + \gamma_1$ results in giving the minimum value $\min E[J^*]$. Recalling $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, and using (18) and (19) we obtain

$$\begin{aligned}&\mathrm{tr} S_1(\Sigma_0 + \boldsymbol{m}_0 \boldsymbol{m}_0^T) \\ &= \mathrm{tr}(\Sigma_0 + \boldsymbol{m}_0 \boldsymbol{m}_0^T)^{1/2T} S_1 (\Sigma_0 + \boldsymbol{m}_0 \boldsymbol{m}_0^T)^{1/2} \\ &= \mathrm{tr}\begin{bmatrix} \boldsymbol{v}_1^T S_1 \boldsymbol{v}_1 & & * \\ & \boldsymbol{v}_2^T S_1 \boldsymbol{v}_2 & \\ & & \ddots & \\ * & & & \boldsymbol{v}_n^T S_1 \boldsymbol{v}_n \end{bmatrix} \\ &= \boldsymbol{v}_1^T S_1 \boldsymbol{v}_1 + \boldsymbol{v}_2^T S_1 \boldsymbol{v}_2 + \cdots + \boldsymbol{v}_n^T S_1 \boldsymbol{v}_n\end{aligned} \tag{50}$$

$$\begin{aligned}&\mathrm{tr} S_2 T\Sigma_0 T^T \\ &= \mathrm{tr} \Sigma_0^{1/2^T} T^T S_2 T \Sigma_0^{1/2}\end{aligned}$$

$$= \mathrm{tr} \begin{bmatrix} \boldsymbol{w}_1^T T^T S_2 T \boldsymbol{w}_1 & & & * \\ & \boldsymbol{w}_2^T T^T S_2 T \boldsymbol{w}_2 & & \\ & & \ddots & \\ * & & & \boldsymbol{w}_n^T T^T S_2 T \boldsymbol{w}_n \end{bmatrix}$$
$$= \boldsymbol{w}_1^T T^T S_2 T \boldsymbol{w}_1 + \cdots + \boldsymbol{w}_n^T T^T S_2 T \boldsymbol{w}_n \qquad (51)$$

Substituing (50) into (48) derives (16)□ Further, combining (51) and (49), and applying Schur Complement leads to (17)□ □

It is noted that the inequalities (13)-(17) cannot be solved directly by LMI because they contain the matrices $S_1^{-1} = S_{1,inv}$, $S_2^{-1} = S_{2,inv}$, $\epsilon_1^{-1} = \epsilon_{1,inv}$, $\epsilon_2^{-1} = \epsilon_{2,inv}$, $\delta_5^{-1} = \delta_{5,inv}$, $\cdots$, $\delta_8^{-1} = \delta_{8,inv}$, $\delta_{13}^{-1} = \delta_{13,inv}$, $\cdots$, $\delta_{16}^{-1} = \delta_{16,inv}$, $\mu_6^{-1} = \mu_{6,inv}$, which satisfy the inverse relations. In addition, the nonlinear terms also appear. Therefore, we apply the cone complementarity linearization approach [12], [13] as well as [11].

4. A NUMERICAL EXAMPLE

Consider a continuous-time system with the following transfer function

$$G(s) = \frac{1+\beta}{s(s+1+\alpha)}$$

where α and β are the uncertainties.

We can convert it to the state space form as follows.

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -1-\alpha & 0 \\ 1 & 0 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 1+\beta \\ 0 \end{bmatrix} \boldsymbol{u}(t)$$
$$\boldsymbol{y}(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \boldsymbol{x}(t)$$

Then, the discrete-time form is

$$\boldsymbol{x}(k+1) = \left\{ \begin{bmatrix} 1-T_s & 0 \\ T_s & 1 \end{bmatrix} + \begin{bmatrix} \Delta a_1 & 0 \\ \Delta a_2 & 0 \end{bmatrix} \right\} \boldsymbol{x}(k) + \left\{ \begin{bmatrix} T_s \\ 0 \end{bmatrix} + \begin{bmatrix} \Delta b_1 \\ \Delta b_2 \end{bmatrix} \right\} \boldsymbol{u}(k)$$
$$\boldsymbol{y}(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} \boldsymbol{x}(k)$$

with $T_s = 0.1$, $\Delta a_1 = 0.006$, $\Delta a_2 = -0.006$, $\Delta b_1 = -0.007$, $\Delta b_2 = 0.006$.

The following parameters are given

$$\boldsymbol{m}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \Sigma_0 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$
$$R = 0.01, \ Q = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}.$$

Applying Theorem 1 with initial γ=2, we obtain a solution

$$K = \begin{bmatrix} -0.0320 & -0.3021 \\ -0.0000 & -1.5397 \end{bmatrix}$$
$$L = -0.3817$$
$$E\left[J^*\right] = 0.3155$$
$$H_0 = 0.0862, \ H_1 = 0.1000, \ E = -0.1002$$
$$W = \begin{bmatrix} 0.3817 \\ 1.0000 \end{bmatrix}$$

Figure 1 displays the transition of the guaranteed cost. The mark + shows that a feasible solution cannot be obtained for the guaranteed cost candidate $\bar{\gamma}$ and the optimal performance index is greater than +. Trajectories of states and estimate error are depicted in Figs. 2-3 with $x(0) = [-0.1 \ \ 0.2]^T$. The control input is illustrated in Fig. 4

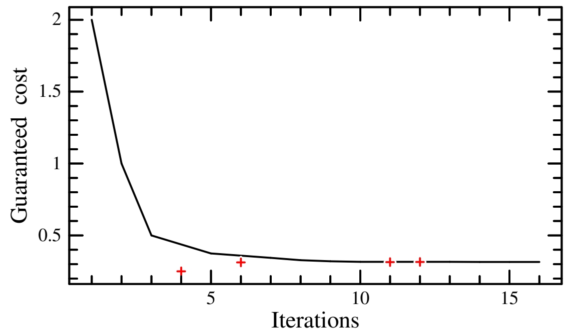

Fig. 1. Trajectories of the guaranteed cost.

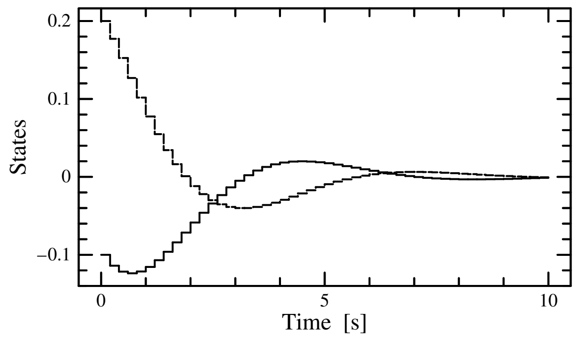

Fig. 2. Trajectories of states x_1 (–) and x_2 (□□).

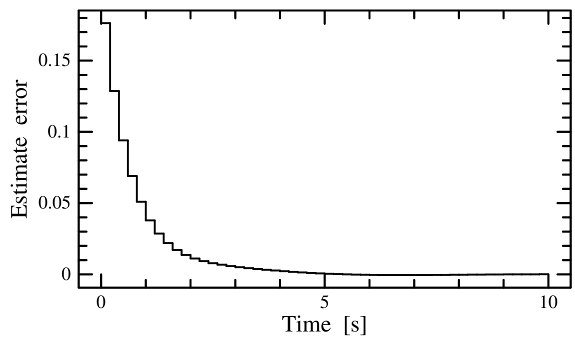

Fig. 3. Trajectory of estimate error ξ.

5. CONCLUSION

This paper discusses a minimal order observer-based guaranteed cost control design for multirate systems. A sufficient condition for the existence of state feedback guaranteed cost controllers is derived on the basis of the LMI feasible solutions.

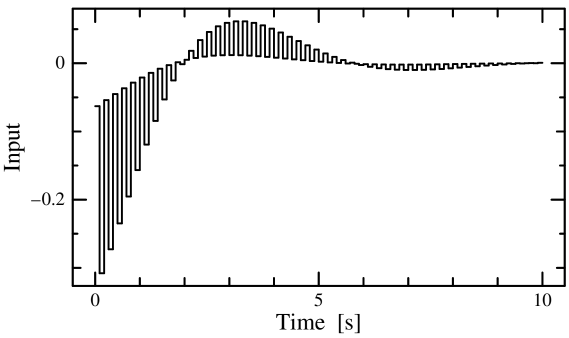

Fig. 4. Trajectory of input u.

REFERENCES

[1] S. S. L. Chang and T. K. C. Peng, "Adaptive guaranteed cost control of systems with uncertain parameters", *IEEE Trans. on Automat. Contr.*, Vol. 17, No. 4, pp. 474–483, 1972.

[2] I. R. Petersen and D. C. McFarlane, "Optimal guaranteed cost control and filtering for uncertain linear systems", *IEEE Trans. on Automat. Contr.*, Vol. 39, No. 9, pp. 1971–1977, 1994.

[3] I. R. Petersen, "Guaranteed cost LQG control of uncertain linear systems", *IEE Proc. Control Theory Appl.*, Vol. 142, No. 2, pp. 95–102, 1995.

[4] L. Xie and Y. C. Soh, "Guaranteed cost control of uncertain discrete-time systems", *Proc. of the 32nd IEEE Conference on Decision and Control*, pp. 56–61, 1993.

[5] L. Yu, J. C. Wang and J. Chu, "Guaranteed cost control of uncertain linear discrete-time systems", *Proc. of the American Control Conference*, pp. 3181–3184, 1997.

[6] I. R. Petersen, D. C. McFarlane and M. A. Rotea, "Optimal guaranteed cost control of discrete-time uncertain linear systems", *Int. J. Robust Nonlinear Control*, Vol. 8, No. 7, pp. 649–657, 1998.

[7] M. Araki and K. Yamamoto, "Multivariable multirate sampled-data systems, state-space description, transfer characteristics and nyquist criterion", *IEEE Trans. on Automat. Contr.*, Vol. 31, No. 2, pp. 145–154, 1986.

[8] J. Salt and P. Albertos, "Model-based multirate controllers design", *IEEE Trans. on Contr. System Technology*, Vol. 13, No. 6, pp. 988–997, 2005.

[9] M. Ishitobi and A. Inoue, "Multirate adaptive control based on a fast-rate model", *Proc. of the 4th International DCDIS Conference.*, pp. 197–201, 2005.

[10] M. Cimino and P. R. Pagilla, "Design of linear time-invariant controllers for multirate systems", *Automatica*, Vol. 46, No. 8, pp. 1315–1319, 2010.

[11] M. Ishitobi, E. Susanto and S. Kunimatsu "Guaranteed cost controller design for multirate systems", *Proc. of the SICE Annual Conference 2012*, pp. 1603–1608, 2012.

[12] L. E. Ghaoui, F. Oustry and M. AitRami, "A cone complementarity linearization algorithm for static output-feedback and related problems", *IEEE Trans. on Automat. Contr.*, Vol. 42, No. 8, pp. 1171–1176, 1997.

[13] W. H. Chen, Z. H. Guan and X. Lu, "Delay-dependent output feedback guaranteed cost control for uncertain time-delay systems", *Automatica*, Vol. 40, No. 7, pp. 1263–1268, 2004.

[14] M. S. Mahmoud and M. Zribi, "Guaranteed cost observer-based control of uncertain time-lag systems", *Computers and Electrical Engineering*, Vol. 29, No. 1, pp. 193–212, 2003.

[15] T. Tamura, "LMI design of a guaranteed cost controller for multirate systems", Master Thesis, Dept. of Mechanical Systems Eng., Kumamoto Univ., Japan (in Japanese).