*IMA Journal of Mathematical Control and Information* (2012) **29**, 113−132
doi:10.1093/imamci/dnr029
Advance Access publication on November 20, 2011

# A minimal-order observer-based guaranteed cost controller for uncertain time-varying delay systems

ERWIN SUSANTO, MITSUAKI ISHITOBI*, SADAAKI KUNIMATSU AND DAISUKE MATSUNAGA
*Department of Mechanical Systems Engineering, Kumamoto University, Kumamoto 860-8555, Japan*
*Corresponding author: mishi@kumamoto-u.ac.jp

[Received on 7 April 2011; revised on 25 July 2011; accepted on 3 September 2011]

This paper considers a design scheme of a minimal-order observer-based guaranteed cost controller for linear uncertain time-varying delay systems. A sufficient condition for asymptotic stabilization is guaranteed via linear matrix inequality feasible solutions. Optimization is provided by minimizing an upper bound of the guaranteed cost function. To show the advantage of a minimal-order observer-based guaranteed cost control approach, a full-order observer-based case is presented as comparison. A numerical example is given to illustrate the proposed method.

*Keywords*: a minimal-order observer; guaranteed cost control; time-varying delay systems; LMI.

## 1. Introduction

It is well known that in many practical control systems, the system model often contains uncertainties and perturbations (Lien, 2005). The presence of the uncertainties and time delay may cause instability and bad performance on a controlled system. Therefore, much effort has been strived to the robust stability and stabilization of the feedback control system with time delay. Especially for the control of the real plant, it is desirable to design a controller which not only achieves the stability of the uncertain system but also guarantees an adequate level of performance (Yu & Chu, 1999). One of the approaches to solve this problem is the guaranteed cost control method (Chang & Peng, 1972) which has an advantage in providing an upper bound on the quadratic cost function of the closed-loop system. This upper bound guarantees that the decrease of the system performance due to uncertainties will not exceed the bound.

Many studies have been reported with significant results based on the guaranteed cost control for uncertain systems via linear matrix inequalities (LMIs), see, e.g. Mahmoud (2001). Although Riccati equation can solve this kind of problems, it needs some additional requirements and limitations to solve specific problems (Yu & Chu, 1999). Furthermore, Boyd *et al.* (1994) showed that many complex problems in system and control theory can be simplified by LMIs form.

In many cases, full states of the system cannot be measured directly because of some reasons such as poor plant knowledge, costing problems, availability, etc. Hence, an observer-based control may be more suitable than a state feedback control in such situation. Mahmoud & Zribi (2003) developed guaranteed cost observer-based controllers for uncertain time-lag systems via solutions of LMIs, though the cost function is not optimized. Recently, Yu & Lien (2007) considered delay-dependent guaranteed cost observer-based control problems for uncertain neutral systems with time-varying delay. However, the convex optimization problem contains the equality constraint. More recently, Ishitobi & Miyachi (2008) investigated an observer-based guaranteed cost control for uncertain systems, but the obtained observer gain is extremely large and some poles are located far in the left half-plane.

Nevertheless, all the previous results considered a full-order observer-based guaranteed cost controller. To the best of our knowledge, no results on the observer-based guaranteed cost control with a minimal-order observer are available in the past. This motivates us to concern a problem on minimal-order observer-based guaranteed cost control. In some full-order observer-based cases, the restrictions on the observer gain are imposed to solve the problem, see, e.g. Mahmoud & Zribi (2003) and Ishitobi & Miyachi (2008). Moreover, an equality constraint and a conservative constraint of time derivative of the time-varying delay appear in Yu & Lien (2007). In our minimal-order observer-based case, optimizing the observer gain leads to obtain less conservative result. Otherwise, a full-order observer-based case yields an extremely large observer gain because it applies no restriction and optimization on the observer gain.

The contribution of this paper is to show a design method of a minimal-order observer-based guaranteed cost controller for uncertain systems with time-varying delay via an LMI technique. As comparison, a full-order observer-based controller design is also presented and it is shown by an example that it yields an extremely large observer gain. On the minimal-order observer design, no restriction is imposed to the observer gain. Optimizing an upper bound of the guaranteed cost is considered and an iterative algorithm is employed to solve the inverse relations. This proposed algorithm utilizes an arbitrary increment. It leads to the faster computation than the algorithm of Chen *et al.* (2004) that applied the decreasing of an initial value to some extent. Similar to that of a full-order observer-based case, it is possible to relax the time derivative of time-varying delay in a minimal-order observer-based problem. It means that a design of a minimal-order observer-based guaranteed cost controller is possible for both a fast and a slow time-varying delay, see, e.g. He *et al.* (2007).

Outline of this paper is as follows. Section 2 states the problem of a continuous uncertain time-varying delay system. Section 3 provides the main results in LMIs, involving a minimal-order observer-based controller design algorithm. Section 4 presents relaxing of time derivative of the time-varying delay. The obtained result is relevant to delay-dependent stabilization. Section 5 shows a full-order observer-based guaranteed cost control problem. Section 6 gives a numerical example to verify the proposed method, and Section 7 concludes this work.

## 2. Problem statement

Consider a continuous uncertain time-varying delay system of the form

$$\dot{x}(t) = (A + \Delta A(t))x(t) + (A_d + \Delta A_d(t))x(t - h(t)) + (B + \Delta B(t))u(t), \tag{2.1}$$

$$y(t) = Cx(t), \tag{2.2}$$

$$x(t) = \psi(t), \quad t \in [-h, 0], \tag{2.3}$$

where $0 \leqslant h(t) \leqslant h$, $\dot{h}(t) \leqslant d < 1$, $h$ is the constant time delay factor in the states and assumed to be known, $d$ is a given value, $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re^r$ is the control input vector, $y(t) \in \Re^m$ is the measured output vector, $x(t) = \psi(t)$ is a given continuous vector-valued initial function, matrices $A$, $A_d$ and $B$ are known constant real-valued matrices with appropriate dimensions and $C$ is restricted to the form of $C = [O\, I_m]$. Matrices $\Delta A(t)$, $\Delta A_d(t)$ and $\Delta B(t)$ denote real-valued matrix functions representing parameter uncertainties.

Remark 2.1 Time derivative of the time-varying delay $\dot{h}(t)$ should be less than one, thus it has a conservative constraint. Taking the advantage of He *et al.* (2007) by using some free-weighting matrices, relaxing $\dot{h}(t)$ can be done such that it can be applied for both a fast and a slow time-varying delay problems.

It is assumed that the initial state variable $x(0)$ is unknown, but their mean and covariance are known, respectively, as

$$E[x(0)] = m_0, \tag{2.4}$$

$$E[(x(0) - m_0)(x(0) - m_0)^\top] = \Sigma_0 > O, \tag{2.5}$$

where $E[\cdot]$ denotes the expected value operator.

The problem considered here is to design a minimal-order observer

$$\dot{z}(t) = Dz(t) + E_m y(t) + Fu(t), \tag{2.6}$$

$$\hat{x}(t) = Pz(t) + Wy(t) \tag{2.7}$$

and a feedback controller

$$u(t) = K\hat{x}(t) \tag{2.8}$$

with

$$D = A_{11} + LA_{21}, \quad PT + WC = I_n, \quad F = TB,$$

$$TA - DT = E_m C, \quad A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right], \quad P = [\, I_{n-m} \quad 0\,]^\top, \quad T = [\, I_{n-m} \quad L\,],$$

where $D, E_m, F, W, A_{11}, A_{12}, A_{21}, A_{22}$ and $L$ are $(n-m) \times (n-m)$, $(n-m) \times m$, $(n-m) \times r$, $n \times m$, $(n-m) \times (n-m)$, $(n-m) \times m$, $m \times (n-m)$, $m \times m$ and $(n-m) \times m$ matrices, respectively, so as to achieve an upper bound on the following quadratic performance index:

$$E[J] = E\left[\int_0^\infty (x^\top(t)Qx(t) + u^\top(t)Ru(t))\mathrm{d}t\right] \tag{2.9}$$

associated with the uncertain time delay system (2.1–2.3), where $Q$ and $R$ are given symmetric positive-definite matrices such that the closed-loop system is asymptotically stable for any time-varying delay $h(t)$.

In the sequel, the following assumptions and facts are needed.

ASSUMPTIONS 2.1 The parameter uncertainties $\Delta A(t)$, $\Delta A_d(t)$ and $\Delta B(t)$ satisfy the following relations:

$$\Delta A(t) = D_A F_A(t) E_A, \quad \Delta A_d(t) = D_{Ad} F_{Ad}(t) E_{Ad}, \quad \Delta B(t) = D_B F_B(t) E_B, \tag{2.10}$$

where $F_A(t)$, $F_{Ad}(t)$ and $F_B(t)$ are unknown time-varying and deterministic matrices satisfying

$$F_A^\top(t) F_A(t) \leqslant I, \quad F_{Ad}^\top(t) F_{Ad}(t) \leqslant I, \quad F_B^\top(t) F_B(t) \leqslant I, \tag{2.11}$$

$D_A$, $D_{Ad}$, $D_B$, $E_A$, $E_{Ad}$ and $E_B$ are constant real-valued known matrices with appropriate dimensions.

.

ASSUMPTIONS 2.2 For $t \in [-h, 0]$, the following relation holds:

$$z(t) - Tm(t) = 0. \tag{2.12}$$

FACT 2.1 (Schur complements, Mahmoud, 2001). Given constant matrices $\Omega_1$, $\Omega_2$ and $\Omega_3$, where $\Omega_1 = \Omega_1^\top$ and $0 < \Omega_2 = \Omega_2^\top$, then $\Omega_1 + \Omega_3^\top \Omega_2^{-1} \Omega_3 < 0$ if and only if

$$\begin{bmatrix} \Omega_1 & \Omega_3^\top \\ \Omega_3 & -\Omega_2 \end{bmatrix} < 0 \quad \text{or} \quad \begin{bmatrix} -\Omega_2 & \Omega_3 \\ \Omega_3^\top & \Omega_1 \end{bmatrix} < 0.$$

FACT 2.2 (Lien, 2005). Given any real matrices $D$ and $E$ with appropriate dimensions, and $F(t)$ be a matrix function satisfying $F(t)^\top F(t) \leqslant I$, then for any positive scalar $\alpha$, the following inequality holds:

$$DF(t)E + E^\top F(t)^\top D^\top \leqslant \alpha D D^\top + \alpha^{-1} E^\top E.$$

## 3. Main results

In this section, a sufficient condition is established for the existence of a minimal-order observer-based guaranteed cost controller for the uncertain time-varying delay system (2.1–2.3). The main result of this study is given by Theorem 3.1. Here, the feedback gain is restricted to the form

$$K = -R^{-1} B^\top S_1, \tag{3.1}$$

where $S_1$ is a symmetric positive-definite matrix.

THEOREM 3.1 Under Assumption 2.1, if the following matrix inequalities optimization problem: $\min\{\gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \mathrm{tr}(M_1) + \mathrm{tr}(M_2)\}$ subject to

$$\begin{bmatrix}
\Lambda_0 & X^\top & X^\top & G_7^\top & A_d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
X & -M_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
X & 0 & -Q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
G_7 & 0 & 0 & -(\zeta+\omega)I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
A_d^\top & 0 & 0 & 0 & \Lambda_1 & G_6^\top & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & G_6 & \Lambda_2 & G_1^\top & G_2^\top & G_3^\top & G_4^\top & G_5^\top & 0 \\
0 & 0 & 0 & 0 & 0 & G_1 & -R & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & G_2 & 0 & -\bar{\omega} I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & G_3 & 0 & 0 & -(\bar{\beta}+\bar{\mu}+\bar{\tau})I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & G_4 & 0 & 0 & 0 & -(\delta+\mu+\tau)I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & G_5 & 0 & 0 & 0 & 0 & -\bar{\lambda} I & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_3
\end{bmatrix} < 0, \tag{3.2}$$

$$\begin{aligned}
&\sum_{k=1}^{n} e_{nk}^\top \Theta_0 e_{nk} < \gamma_0, \quad \sum_{k=1}^{m} e_{mk}^\top \Theta_1 e_{mk} < \gamma_1, \\
&\sum_{k=1}^{m} e_{mk}^\top \Theta_2 e_{mk} < \gamma_2, \quad \sum_{k=1}^{m} e_{mk}^\top \Theta_3 e_{mk} < \gamma_3,
\end{aligned} \tag{3.3}$$

$$\begin{bmatrix} -\gamma_4 & \upsilon_1^\top Y^\top & \upsilon_2^\top Y^\top & \cdots & \upsilon_m^\top Y^\top \\ Y\upsilon_1 & -S_2 & & & \vdots \\ Y\upsilon_2 & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ Y\upsilon_m & \cdots & \cdots & \cdots & -S_2 \end{bmatrix} < 0, \tag{3.4}$$

$$\begin{bmatrix} -M_1 & E_x \\ E_x^\top & -S_3^{-1} \end{bmatrix} < 0, \tag{3.5}$$

$$\begin{bmatrix} -M_2 & F_x \\ F_x^\top & -S_4^{-1} \end{bmatrix} < 0, \tag{3.6}$$

where

$$\begin{aligned}
\Lambda_0 &= AX + XA^\top - BR^{-1}B^\top + \zeta D_A D_A^\top + (\epsilon+\delta) D_B D_B^\top \\
&\quad + (\bar{\epsilon}+\bar{\beta}) BR^{-1}E_B^\top E_B R^{-1} B^\top + \theta D_{Ad} D_{Ad}^\top, \\
\Lambda_1 &= -(1-d)S_3 + (\bar{\theta}+\bar{\mu}) E_{Ad} E_{Ad}^\top, \quad \Lambda_2 = S_2 A_{11} + A_{11}^\top S_2 + YA_{21} + A_{21}^\top Y^\top + S_4, \\
\Lambda_3 &= -(1-d)S_4, \\
Y &= S_2 L, \quad Z = [S_2 Y], \\
G_1 &= P^\top S_1 B, \quad G_2 = D_A^\top Z^\top, \quad G_3 = D_B^\top Z^\top, \quad G_4 = E_B R^{-1} B^\top S_1 P, \\
G_5 &= D_{Ad}^\top Z^\top, \quad G_6 = -Z A_d, \quad G_7 = E_A X, \\
E_x &= hI_n, \quad F_x = hI_{n-m}, \quad \Theta_0 = \frac{1}{2}(S_1(\Sigma_0 + m_0 m_0^\top) + (\Sigma_0 + m_0 m_0^\top)^\top S_1), \\
\Theta_1 &= \frac{1}{2}(S_2 \Sigma_{11} + \Sigma_{11} S_2), \\
\Theta_2 &= \frac{1}{2}(Y\Sigma_{21} + \Sigma_{21}^\top Y^\top), \quad \Theta_3 = \frac{1}{2}(Y^\top \Sigma_{12} + \Sigma_{12}^\top Y), \quad \Sigma_{22}^{1/2} = [\upsilon_1, \upsilon_2, \ldots, \upsilon_m], \\
\Sigma_0 &= \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}, \quad e_{ik} = [0_{k-1}^\top \quad 1 \quad 0_{i-k}^\top]^\top,
\end{aligned}$$

where $\Sigma_0$, $\Sigma_{11}$, $\Sigma_{12}$, $\Sigma_{21}$ and $\Sigma_{22}$ are $n \times n$, $(n-m) \times (n-m)$, $(n-m) \times m$, $m \times (n-m)$ and $m \times m$ matrices, respectively, has a set of solutions $S_1 > 0$, $S_2 > 0$, $S_3 > 0$, $S_4 > 0$, $M_x > 0$, $X > 0$, $Y$, $Z$, $\zeta > 0$, $\epsilon > 0$, $\bar{\epsilon} > 0$, $\theta > 0$, $\bar{\theta} > 0$, $\bar{\beta} > 0$, $\delta > 0$, $\omega > 0$, $\bar{\omega} > 0$, $\bar{\lambda} > 0$, $\mu > 0$, $\bar{\mu} > 0$, $\tau > 0$, $\bar{\tau} > 0$, $\gamma_0$, $\gamma_1$, $\gamma_2$, $\gamma_3$ and $\gamma_4$ which satisfy the relation $\epsilon^{-1} = \bar{\epsilon}$, $\theta^{-1} = \bar{\theta}$, $\omega^{-1} = \bar{\omega}$, $\mu^{-1} = \bar{\mu}$, $\tau^{-1} = \bar{\tau}$, $S_1^{-1} = X$ and $S_3^{-1} = M_x$, then the minimal-order observer-based control law (2.6–2.8) with $K = -R^{-1}B^\top S_1$ and $L = S_2^{-1}Y$ is a guaranteed cost controller which gives the minimum expected value of the guaranteed cost

$$\begin{aligned}
E[J^*] &= E\left[ w^\top(0) \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} w(0) + \int_{-h(0)}^{0} w^\top(s) \begin{bmatrix} S_3 & 0 \\ 0 & S_4 \end{bmatrix} w(s) \mathrm{d}s \right] \\
&= \min \{\gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \mathrm{tr}(M_1) + \mathrm{tr}(M_2)\},
\end{aligned} \tag{3.7}$$

where $w(\cdot) = [x(\cdot)^\top \quad \xi(\cdot)^\top]^\top$ and $\xi(t) = z(t) - Tx(t)$ is the estimated error of the minimal-order observer.

REMARK 3.1 Since (3.2) has a constraint of the inverse relations, an iterative algorithm via LMI approach is introduced to solve El Ghaoui *et al.* (1997).

*Proof.* Equations (2.1–2.3) and (2.6–2.8) yield the closed-loop system

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\xi}(t) \end{bmatrix} = \begin{bmatrix} \Phi_1 & \Phi_2 & \Phi_3 \\ \Phi_4 & \Phi_5 & \Phi_6 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \\ \xi(t) \end{bmatrix}, \tag{3.8}$$

where

$$\Phi_1 = A + \varDelta A(t) + (B + \varDelta B(t))K, \quad \Phi_2 = A_d + \varDelta A_d(t), \quad \Phi_3 = (B + \varDelta B(t))KP,$$
$$\Phi_4 = -T\varDelta A(t) - T\varDelta B(t)K, \quad \Phi_5 = -T(A_d + \varDelta A_d(t)), \quad \Phi_6 = D - T\varDelta B(t)KP.$$

Further, consider a candidate of Lyapunov function as follows:

$$V(t) = V_1(t) + V_2(t), \tag{3.9}$$

where

$$V_1(t) = w^\top(t) \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} w(t), \quad V_2(t) = \int_{t-h(t)}^{\top} w^\top(t) \begin{bmatrix} S_3 & 0 \\ 0 & S_4 \end{bmatrix} w(t)\mathrm{d}t.$$

Then the time derivative of (3.9) along with (3.8) is calculated in the following equation:

$$\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t), \tag{3.10}$$

where $\dot{V}_1(t)$ and $\dot{V}_2(t)$ are

$$\dot{V}_1(t) = \dot{w}^\top(t) \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} w(t) + w^\top(t) \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \dot{w}(t),$$

$$\begin{aligned} \dot{V}_2(t) &= w^\top(t) \begin{bmatrix} S_3 & 0 \\ 0 & S_4 \end{bmatrix} w(t) - (1 - \dot{h}(t))w^\top(t-h(t)) \begin{bmatrix} S_3 & 0 \\ 0 & S_4 \end{bmatrix} w(t-h(t)) \\ &\leqslant w^\top(t) \begin{bmatrix} S_3 & 0 \\ 0 & S_4 \end{bmatrix} w(t) - (1-d)w^\top(t-h(t)) \begin{bmatrix} S_3 & 0 \\ 0 & S_4 \end{bmatrix} w(t-h(t)), \end{aligned}$$

respectively.

By introducing $\bar{z}(t)$ and $\Omega(t)$, (3.10) can be written as

$$\dot{V}(t) = \bar{z}^\top(t)\Omega(t)\bar{z}(t) - (x^\top(t)Qx(t) + u^\top(t)Ru(t)), \tag{3.11}$$

$$\bar{z}(t) = \begin{bmatrix} x(t) \\ x(t-h(t)) \\ \xi(t) \\ \xi(t-h(t)) \end{bmatrix}, \quad \Omega(t) = \begin{bmatrix} \Omega_1 & \Omega_2 & \Omega_3 & 0 \\ \Omega_2^\top & \Omega_4 & \Omega_5 & 0 \\ \Omega_3^\top & \Omega_5^\top & \Omega_6 & 0 \\ 0 & 0 & 0 & \Omega_7 \end{bmatrix},$$

$$\begin{aligned}
\varOmega_1 &= S_1(A + \varDelta A(t) + (B + \varDelta B(t))K) + (A + \varDelta A(t) \\
&\quad + (B + \varDelta B(t))K)^\top S_1 + S_3 + Q + K^\top RK, \\
\varOmega_2 &= S_1(A_d + \varDelta A_d(t)), \\
\varOmega_3 &= -\varDelta A^\top(t)T^\top S_2 - K^\top \varDelta B^\top(t)T^\top S_2 - S_1\varDelta B(t)R^{-1}B^\top S_1P, \\
\varOmega_4 &= -(1 - \dot{h}(t))S_3, \\
\varOmega_5 &= -(A_d + \varDelta A_d(t))^\top T^\top S_2, \\
\varOmega_6 &= S_2D + D^\top S_2 - S_2T\varDelta B(t)KP - P^\top K^\top \varDelta B^\top(t)T^\top S_2 \\
&\quad + S_4 + P^\top S_1BR^{-1}B^\top S_1P, \\
\varOmega_7 &= -(1 - \dot{h}(t))S_4.
\end{aligned}$$

Under the condition

$$\varOmega(t) < 0, \tag{3.12}$$

(3.11) leads to

$$\dot{V}(t) < -(x^\top(t)Qx(t) + u^\top(t)Ru(t)) < 0, \tag{3.13}$$

for any $x(t) \neq 0$ and the closed-loop system is asymptotically stable.

Here, (3.12) is investigated below. By applying Fact 2.2 to (3.12), it follows for any $\zeta > 0$, $\epsilon > 0$, $\theta > 0$, $\omega > 0$, $\beta > 0$, $\delta > 0$, $\lambda > 0$, $\mu > 0$ and $\tau > 0$ that

$$\begin{aligned}
2x^\top(t)S_1\varDelta A(t)x(t) &= 2x^\top(t)S_1D_AF_A(t)E_Ax(t) \\
&\leqslant \zeta x^\top(t)S_1D_AD_A^\top S_1x(t) + \zeta^{-1}x^\top(t)E_A^\top E_Ax(t),
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
-2x^\top(t)S_1\varDelta B(t)R^{-1}B^\top S_1x(t) &= -2x^\top(t)S_1D_BF_B(t)E_BR^{-1}B^\top S_1x(t) \\
&\leqslant \epsilon x^\top(t)S_1D_BD_B^\top S_1x(t) \\
&\quad + \epsilon^{-1}x^\top(t)S_1BR^{-1}E_B^\top E_BR^{-1}B^\top S_1x(t),
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
2x^\top(t)S_1\varDelta A_d(t)x(t-h(t)) &= 2x^\top(t)S_1D_{Ad}F_{Ad}(t)E_{Ad}x(t-h(t)) \\
&\leqslant \theta x^\top(t)S_1D_{Ad}D_{Ad}^\top S_1x(t) \\
&\quad + \theta^{-1}x^\top(t-h(t))E_{Ad}^\top E_{Ad}x(t-h(t)),
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
-2\xi^\top(t)S_2T\varDelta A(t)x(t) &= -2\xi^\top(t)S_2TD_AF_A(t)E_Ax(t) \\
&\leqslant \omega\xi^\top(t)S_2TD_AD_A^\top T^\top S_2\xi(t) \\
&\quad + \omega^{-1}x^\top(t)E_A^\top E_Ax(t),
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
2x^\top(t)S_1BR^{-1}\varDelta B^\top(t)T^\top S_2\xi(t) &= 2\xi^\top(t)S_2TD_BF_B(t)E_BR^{-1}B^\top S_1x(t) \\
&\leqslant \beta\xi^\top(t)S_2TD_BD_B^\top T^\top S_2\xi(t) \\
&\quad + \beta^{-1}x^\top(t)S_1BR^{-1}E_B^\top E_BR^{-1}B^\top S_1x(t),
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
-2x^\top(t)S_1\varDelta B(t)R^{-1}B^\top S_1P\xi(t) &= -2x^\top(t)S_1D_BF_B(t)E_BR^{-1}B^\top S_1P\xi(t)\\
&\leqslant \delta x^\top(t)S_1D_BD_B^\top S_1x(t)\\
&\quad+\delta^{-1}\xi^\top(t)P^\top S_1BR^{-1}E_B^\top E_BR^{-1}B^\top S_1P\xi(t),
\end{aligned}\tag{3.19}$$

$$\begin{aligned}
-2x^\top(t-h(t))\varDelta A_d^\top(t)T^\top S_2\xi(t) &= -2\xi^\top(t)S_2TD_{Ad}F_{Ad}(t)E_{Ad}x(t-h(t))\\
&\leqslant \lambda\xi^\top(t)S_2TD_{Ad}D_{Ad}^\top T^\top S_2\xi(t)\\
&\quad+\lambda^{-1}x^\top(t-h(t))E_{Ad}^\top E_{Ad}x(t-h(t)),
\end{aligned}\tag{3.20}$$

$$\begin{aligned}
2\xi^\top(t)S_2T\varDelta B(t)R^{-1}B^\top S_1P\xi(t) &= 2\xi^\top(t)S_2TD_BF_B(t)E_BR^{-1}B^\top S_1P\xi(t)\\
&\leqslant \mu\xi^\top(t)S_2TD_BD_B^\top T^\top S_2\xi(t)\\
&\quad+\mu^{-1}\xi^\top(t)P^\top S_1BR^{-1}E_B^\top E_BR^{-1}B^\top S_1P\xi(t),
\end{aligned}\tag{3.21}$$

$$\begin{aligned}
2\xi^\top(t)P^\top S_1BR^{-1}\varDelta B^\top(t)T^\top S_2\xi(t) &= 2\xi^\top(t)S_2TD_BF_B(t)E_BR^{-1}B^\top S_1P\xi(t)\\
&\leqslant \tau\xi^\top(t)S_2TD_BD_B^\top T^\top S_2\xi(t)\\
&\quad+\tau^{-1}\xi^\top(t)P^\top S_1BR^{-1}E_B^\top E_BR^{-1}B^\top S_1P\xi(t).
\end{aligned}\tag{3.22}$$

Hence, if there exist positive scalars $\zeta$, $\epsilon$, $\theta$, $\omega$, $\beta$, $\delta$, $\lambda$, $\mu$ and $\tau$ and symmetric positive-definite matrices $S_1$, $S_2$, $S_3$ and $S_4$ which satisfy the following matrix inequality:

$$\begin{bmatrix} \varOmega_8 & \varOmega_9 & 0 & 0\\ \varOmega_9^\top & \varOmega_{10} & \varOmega_{11} & 0\\ 0 & \varOmega_{11}^\top & \varOmega_{12} & 0\\ 0 & 0 & 0 & \varOmega_{13}\end{bmatrix} < 0, \tag{3.23}$$

where

$$\begin{aligned}
\varOmega_8 &= S_1A + A^\top S_1 - S_1BR^{-1}B^\top S_1 + S_3 + Q + \zeta S_1D_AD_A^\top S_1 + (\zeta^{-1}+\omega^{-1})E_A^\top E_A\\
&\quad+(\epsilon+\delta)S_1D_BD_B^\top S_1 + (\epsilon^{-1}+\beta^{-1})S_1BR^{-1}E_B^\top E_BR^{-1}B^\top S_1 + \theta S_1D_{Ad}D_{Ad}^\top S_1,\\
\varOmega_9 &= S_1A_d,\quad \varOmega_{10} = -(1-\dot h(t))S_3 + (\theta^{-1}+\mu^{-1})E_{Ad}^\top E_{Ad},\quad \varOmega_{11} = -A_d^\top T^\top S_2,\\
\varOmega_{12} &= S_2D + D^\top S_2 + S_4 + \omega S_2TD_AD_A^\top T^\top S_2 + (\beta+\mu+\tau)S_2TD_BD_B^\top T^\top S_2\\
&\quad+\lambda S_2TD_{Ad}D_{Ad}^\top T^\top S_2 + (\delta^{-1}\mu^{-1}+\tau^{-1})P^\top S_1BR^{-1}E_B^\top E_BR^{-1}B^\top S_1P,\\
\varOmega_{13} &= -(1-\dot h(t))S_4,
\end{aligned}$$

then (2.8) is a minimal-order observer-based guaranteed cost control law and (3.7) is a guaranteed cost for the uncertain time-varying delay system (2.1–2.3).

Pre- and postmultiplying (3.23) by $\mathrm{diag}(S_1^{-1},\ I,\ I,\ I)$ on both sides, denoting $X = S_1^{-1}$, $M_x = S_3^{-1}$, $Y = S_2L$, $\bar\epsilon = \epsilon^{-1}$, $\bar\theta = \theta^{-1}$, $\bar\omega = \omega^{-1}$, $\bar\mu = \mu^{-1}$, $\bar\tau = \tau^{-1}$, substituting $D = A_{11} + LA_{21}$ into (3.23) and using Schur complement lead to (3.2).

Further, integrating (3.13) from 0 to $T \to \infty$ yields

$$\begin{aligned} J &= \int_0^\infty (x^\top(t)Qx(t) + u^\top(t)Ru(t))\mathrm{d}t \\ &< x^\top(0)S_1x(0) + \xi^\top(0)S_2\xi(0) + \int_{-h(0)}^0 x^\top(s)S_3x(s)\mathrm{d}s + \int_{-h(0)}^0 \xi^\top(s)S_4\xi(s)\mathrm{d}s = J^*, \end{aligned} \tag{3.24}$$

where $J^*$ denotes the guaranteed cost. Here, we consider the optimal expected value of the guaranteed cost. It is calculated as

$$\begin{aligned} E[J^*] = {} & \mathrm{tr}S_1E[x(0)x^\top(0)] + \mathrm{tr}S_2E[\xi(0)\xi^\top(0)] \\ & + \mathrm{tr}S_3E\left[\int_{-h(0)}^0 x(s)x^\top(s)\mathrm{d}s\right] + \mathrm{tr}S_4E\left[\int_{-h(0)}^0 \xi(s)\xi^\top(s)\mathrm{d}s\right]. \end{aligned} \tag{3.25}$$

Substituting (2.5) into (3.25) and using Assumption 2.2 result in

$$\begin{aligned} E[J^*] = {} & \mathrm{tr}S_1(\Sigma_0 + m_0m_0^\top) + \mathrm{tr}S_2E[(z(0) - Tx(0))(z(0) - Tx(0))^\top] \\ & + \mathrm{tr}S_3E\left[\int_{-h(0)}^0 x(s)x^\top(s)\mathrm{d}s\right] + \mathrm{tr}S_4E\left[\int_{-h(0)}^0 \xi(s)\xi^\top(s)\mathrm{d}s\right]. \end{aligned} \tag{3.26}$$

Here, it is readily seen from (3.26) that

$$E[(z(0) - Tx(0))(z(0) - Tx(0))^\top] = T\Sigma_0T^\top + (z(0) - Tm_0)(z(0) - Tm_0)^\top, \tag{3.27}$$

$$\mathrm{tr}S_3E\left[\int_{-h(0)}^0 x(s)x^\top(s)\mathrm{d}s\right] = \mathrm{tr}S_3E_xE_x^\top, \tag{3.28}$$

$$\mathrm{tr}S_4E\left[\int_{-h(0)}^0 \xi(s)\xi^\top(s)\mathrm{d}s\right] = \mathrm{tr}S_4F_xF_x^\top. \tag{3.29}$$

Next, consider positive scalars $\gamma_0$, $\gamma_1$, $\gamma_2$, $\gamma_3$, $\gamma_4$, $\mathrm{tr}(M_1)$ and $\mathrm{tr}(M_2)$ satisfying the following inequalities:

$$\mathrm{tr}S_1(\Sigma_0 + m_0m_0^\top) < \gamma_0, \tag{3.30}$$

$$\mathrm{tr}S_2\Sigma_{11} < \gamma_1, \tag{3.31}$$

$$\mathrm{tr}S_2L\Sigma_{21} < \gamma_2, \tag{3.32}$$

$$\mathrm{tr}S_2\Sigma_{12}L^\top < \gamma_3, \tag{3.33}$$

$$\mathrm{tr}S_2L\Sigma_{22}L^\top < \gamma_4, \tag{3.34}$$

$$\mathrm{tr}S_3E_xE_x^\top < \mathrm{tr}(M_1), \tag{3.35}$$

$$\mathrm{tr}S_4F_xF_x^\top < \mathrm{tr}(M_2). \tag{3.36}$$

Minimizing $\gamma_0+\gamma_1+\gamma_2+\gamma_3+\gamma_4+\mathrm{tr}(M_1)+\mathrm{tr}(M_2)$ results in giving $\min E[J^*]$. By recalling $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, (3.30–3.33) lead to (3.3) and (3.35–3.36) lead to (3.5) and (3.6), respectively. Next, by denoting $\Sigma_{22}^{1/2} = [\upsilon_1, \upsilon_2, \dots, \upsilon_m]$, (3.34) is calculated as

$$\begin{aligned}\mathrm{tr} S_2 L \Sigma_{22} L^\top &= \upsilon_1^\top Y^\top S_2^{-1} Y \upsilon_1 + \upsilon_2^\top Y^\top S_2^{-1} Y \upsilon_2 + \cdots + \upsilon_m^\top Y^\top S_2^{-1} Y \upsilon_m \\ &= [\upsilon_1^\top Y^\top \quad \upsilon_2^\top Y^\top \quad \cdots \quad \upsilon_m^\top Y^\top] S_2^{-1} \begin{bmatrix} Y\upsilon_1 \\ Y\upsilon_2 \\ \vdots \\ Y\upsilon_m \end{bmatrix} < \gamma_4. \end{aligned} \tag{3.37}$$

Further, Schur complement derives (3.4) from (3.37). □

REMARK 3.2 Minimizing the observer gain in (3.32–3.34) is an important key to get less conservative result than that of a full-order observer-based case. Otherwise, restriction on the observer gain form and/or the stabilization formulations are needed, see, e.g. Mahmoud & Zribi (2003), Yu & Lien (2007) and Ishitobi & Miyachi (2008).

It is also noted that the inequalities (3.2) cannot be solved directly by LMI because they contain the scalars $\epsilon$, $\bar\epsilon$, $\theta$, $\bar\theta$, $\omega$, $\bar\omega$ $\mu$, $\bar\mu$, $\tau$ and $\bar\tau$ and matrices $S_1$, $X$, $S_3$ and $M_x$ which satisfy the relation $S_1^{-1}X = I$, $S_3^{-1}M_x = I$, $\epsilon^{-1}\bar\epsilon = 1$, $\theta^{-1}\bar\theta = 1$, $\omega^{-1}\bar\omega = 1$, $\mu^{-1}\bar\mu = 1$ and $\tau^{-1}\bar\tau = 1$. There are a number of algorithms available in literature, a cone complementarity linearization approach (El Ghaoui *et al.*, 1997), a sequential linear programming matrix method (Leibfritz, 2001), a Min–Max algorithm, an alternating projection method and so on to solve this kind of the problems. Here, we apply the cone complementarity linearization approach and propose the algorithm as follows.

ALGORITHM

0: Set $k_{\max}$, $\gamma_{\min}$ and $\kappa$.

1: Choose a sufficiently large initial $\gamma$ such that there exists a feasible solution to LMI conditions

$$\begin{bmatrix} S_1 & I \\ I & X \end{bmatrix} > 0, \quad \begin{bmatrix} S_3 & I \\ I & M_x \end{bmatrix} > 0, \quad \gamma_0+\gamma_1+\gamma_2+\gamma_3+\gamma_4+\mathrm{tr}(M_1)+\mathrm{tr}(M_2) < \gamma,$$
$$\epsilon\bar\epsilon > 1, \quad \theta\bar\theta > 1, \quad \omega\bar\omega > 1, \quad \mu\bar\mu > 1, \quad \tau\bar\tau > 1, \quad \text{inequalities (3.2–3.6).}$$

2.1: Set $\bar\gamma = \gamma$, $k = 0$, $i = 1$, $S_1(k) = S_1$, $X(k) = X$, $S_3(k) = S_3$, $M_x(k) = M_x$, $\epsilon(k) = \epsilon$, $\bar\epsilon(k) = \bar\epsilon$, $\theta(k) = \theta$, $\bar\theta(k) = \bar\theta$, $\omega(k) = \omega$, $\bar\omega(k) = \bar\omega$, $\mu(k) = \mu$, $\bar\mu(k) = \bar\mu$, $\tau(k) = \tau$, $\bar\tau(k) = \bar\tau$.

2.2: Solve the following LMI problem:

$$\begin{aligned} t_k = \mathrm{Min}\{&\mathrm{tr}[S_1(k)X + X(k)S_1 + S_3(k)M_x + M_x(k)S_3] + \epsilon(k)\bar\epsilon + \epsilon\bar\epsilon(k) + \theta(k)\bar\theta + \theta\bar\theta(k) \\ &+\omega(k)\bar\omega + \omega\bar\omega(k) + \mu(k)\bar\mu + \mu\bar\mu(k) + \tau(k)\bar\tau + \tau\bar\tau(k)\} \end{aligned}$$

subject to

$$\begin{bmatrix} S_1 & I \\ I & X \end{bmatrix} > 0, \quad \begin{bmatrix} S_3 & I \\ I & M_x \end{bmatrix} > 0, \quad \gamma_0+\gamma_1+\gamma_2+\gamma_3+\gamma_4+\mathrm{tr}(M_1)+\mathrm{tr}(M_2) < \gamma,$$
$$\epsilon\bar\epsilon > 1, \quad \theta\bar\theta > 1, \quad \omega\bar\omega > 1, \quad \mu\bar\mu > 1, \quad \tau\bar\tau > 1, \quad \text{inequalities (3.2–3.6).}$$

3.1: If $k < k_{\max}$ and $t_k > 4n + 10 + \kappa$, then set $k = k + 1$ and go to 2.2.

3.2: If $k \leqslant k_{\max}$, $4n + 10 < t_k \leqslant 4n + 10 + \kappa$, LMI conditions are satisfied and $\gamma\,(0.5)^i > \gamma_{\min}$, then $\bar{\gamma} = \bar{\gamma} - \gamma\,(0.5)^i$. Else if $\gamma\,(0.5)^i \leqslant \gamma_{\min}$, then exit and $\bar{\gamma}$ is an optimal value.

3.3: If $k < k_{\max}$, $4n + 10 < t_k \leqslant 4n + 10 + \kappa$, LMI conditions are not satisfied, $i \neq 1$ and $\gamma\,(0.5)^i > \gamma_{\min}$, then $\bar{\gamma} = \bar{\gamma} + \gamma\,(0.5)^i$. Else if $\gamma\,(0.5)^i \leqslant \gamma_{\min}$, then exit and $\bar{\gamma}$ is an optimal value. Else if $i = 1$, then exit and no optimal solution is obtained.

3.4: If $k = k_{\max}$, $t_k > 4n + 10 + \kappa$, $i \neq 1$ and $\gamma\,(0.5)^i > \gamma_{\min}$, then $\bar{\gamma} = \bar{\gamma} + \gamma\,(0.5)^i$. Else if $\gamma\,(0.5)^i \leqslant \gamma_{\min}$, then exit and $\bar{\gamma}$ is an optimal value. Else if $i = 1$, then exit and no optimal solution is obtained.

4: Set $i = i + 1$ and return to 3.1.

REMARK 3.3 We utilize an arbitrary increment $\pm\gamma\,(0.5)^i$ that leads this algorithm to the faster computation than algorithm of Chen *et al.* (2004) in finding an optimal value of guaranteed cost.

## 4. The relaxed time derivative of time-varying delay

This section is relevant for delay-dependent stability, while Section 3 considers delay-independent problem. In order to relax a constraint of time derivative of time-varying delay, LMI (3.2) in Theorem 3.1 shall be changed according to He *et al.* (2007) by this inequality

$$\begin{bmatrix} \nabla_1 & X^\top & A_d & 0 & 0 & \nabla_6 & N_1 & N_2^\top - N_1 \\ X & \nabla_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_d^\top & 0 & \nabla_3 & A_d^\top T^\top S_2 & 0 & \nabla_7 & N_2 & 0 \\ 0 & 0 & S_2 T A_d & \nabla_4 & \nabla_5 & \nabla_8 & 0 & 0 \\ 0 & 0 & 0 & \nabla_5^\top & -R & 0 & 0 & 0 \\ \nabla_6^\top & 0 & \nabla_7^\top & \nabla_8^\top & 0 & -(hS_4)^{-1} & 0 & 0 \\ N_1^\top & 0 & N_2^\top & 0 & 0 & 0 & -h^{-1}S_4 & 0 \\ N_2 - N_1^\top & 0 & 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix} + \begin{bmatrix} \Delta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} < 0, \tag{4.1}$$

where

$$\nabla_1 = AX + XA^\top - BR^{-1}B^\top, \quad \nabla_2 = -(N_1^\top + N_1 + S_3 + Q)^{-1},$$
$$\nabla_3 = -(1-\dot{h}(t))S_3 - N_2 - N_2^\top, \nabla_4 = S_2D + D^\top S_2, \quad \nabla_5 = P^\top S_1 B,$$
$$\nabla_6 = (A - BR^{-1}B^\top S_1)^\top, \quad \nabla_7 = A_d^\top, \quad \nabla_8 = P^\top S_1 BR^{-1}B^\top,$$
$$\begin{aligned}\varDelta_1 = {} & (\epsilon_1+\alpha_1)D_A D_A^\top + (\epsilon_2+\epsilon_4+\alpha_2+\alpha_4)D_B D_B^\top + (\epsilon_2^{-1}+\epsilon_6^{-1})BR^{-1}E_B^\top E_B R^{-1}B^\top \\ & + (\epsilon_3+\alpha_3)D_{Ad}D_{Ad}^\top + (\epsilon_1^{-1}+\epsilon_5^{-1})XE_A^\top E_A X + \alpha_2^{-1}S_1BR^{-1}E_B^\top E_B R^{-1}B^\top S_1,\end{aligned}$$
$$\varDelta_2 = (\epsilon_3^{-1}+\epsilon_7^{-1}+\alpha_3^{-1})E_{Ad}^\top E_{Ad},$$
$$\begin{aligned}\varDelta_3 = {} & \epsilon_7 S_2 T D_{Ad} D_{Ad}^\top T^\top S_2 + (\epsilon_6+\epsilon_8)S_2 T D_B D_B^\top T^\top S_2 \\ & + (\epsilon_4^{-1}+\epsilon_8^{-1}+\alpha_4^{-1})P^\top S_1 BR^{-1}E_B^\top E_B R^{-1}B^\top S_1 P + \epsilon_5 S_2 T D_A D_A^\top T^\top S_2.\end{aligned}$$

One can see that $\dot{h}(t)$ or $d$ can be any value since $\nabla_3$ also contains other negative variable matrices even though term $-(1-\dot{h}(t))S_3$ appears in it.

*Proof.* Using Leibniz–Newton formula

$$x(t) = x(t-h(t)) + \int_{t-h(t)}^{t} \dot{x}(s)\mathrm{d}s \tag{4.2}$$

and substituting (4.2) into (2.1), then we get

$$\begin{aligned}\dot{x}(t) = {} & (A + \varDelta A(t) + (B + \varDelta B(t))K + A_d + \varDelta A_d(t))x(t) \\ & - (A_d + \varDelta A_d(t)) \int_{t-h(t)}^{t} \dot{x}(s)\mathrm{d}s + (B + \varDelta B(t))KP\xi(t).\end{aligned} \tag{4.3}$$

Consider the Lyapunov function candidate for delay-dependent stability of uncertain time-varying delay systems

$$\begin{aligned}V(t) = {} & x^\top(t)S_1x(t) + \xi^\top(t)S_2\xi(t) + \int_{t-h(t)}^{t} x^\top(s)S_3x(s)\mathrm{d}s \\ & + \int_{t-h(t)}^{t}\int_{s}^{t} \dot{x}^\top(\theta)S_4\dot{x}(\theta)\mathrm{d}\theta\,\mathrm{d}s.\end{aligned} \tag{4.4}$$

Time derivative of (4.4) is

$$\begin{aligned}\dot{V}(t) = {} & 2x^\top(t)S_1\dot{x}(t) + 2\xi^\top(t)S_2\dot{\xi}(t) + x^\top(t)S_3x(t) - (1-\dot{h}(t))x^\top(t-h(t))S_3x(t-h(t)) \\ & + h\dot{x}^\top(t)S_4\dot{x}(t) - \int_{t-h(t)}^{t} \dot{x}^\top(s)S_4\dot{x}(s)\mathrm{d}s.\end{aligned} \tag{4.5}$$

To obtain a less conservative stabilization criterion, the following term holds:

$$2[x^\top N_1 + x^\top(t-h(t))N_2] \times \left[x(t) - x(t-h(t)) - \int_{t-h(t)}^{t} \dot{x}(s)\mathrm{d}s\right] = 0, \tag{4.6}$$

where $N_1$ and $N_2$ are free-weighting appropriate dimensioned matrices.

Adding (4.6) to (4.5) and considering a similar method to (3.11), we obtain a new upper bound of (4.5):

$$\begin{aligned}\dot{V}(t) \leqslant\ & 2x^\top(t)S_1[(A+\varDelta A(t)+(B+\varDelta B(t))K)x(t)+(A_d+\varDelta A_d(t))x(t-h(t))+(B+\varDelta B(t))KP\xi(t)]\\ &+2\xi^\top(t)S_2[-T(\varDelta A(t)+\varDelta B(t)K)x(t)-T(A_d+\varDelta A_d(t))x(t-h(t))+(D-T\varDelta B(t)KP)\xi(t)]\\ &+x^\top(t)S_3x(t)-(1-\dot{h})x(t-h(t))S_3x(t-h(t))\\ &+h[(A+\varDelta A(t)+(B+\varDelta B(t))K)x(t)+(A_d+\varDelta A_d(t))x(t-h(t))+(B+\varDelta B(t))KP\xi(t)]^\top S_4\\ &\times[(A+\varDelta A(t)+(B+\varDelta B(t))K)x(t)+(A_d+\varDelta A_d(t))x(t-h(t))+(B+\varDelta B(t))KP\xi(t)]\\ &-\int_{t-h(t)}^{t}\dot{x}^\top(s)S_4\dot{x}(s)\mathrm{d}s+2[x^\top(t)N_1+x^\top(t-h(t))N_2]\times\left[x(t)-x(t-h(t))-\int_{t-h(t)}^{t}\dot{x}(s)\mathrm{d}s\right]\\ &+x^\top(t)Qx(t)+(x(t)+P\xi(t))^\top K^\top RK(x(t)+P\xi(t))-[x^\top(t)Qx(t)+u^\top(t)Ru(t)].\end{aligned}\tag{4.7}$$

We can note from (4.7) that

$$\begin{aligned}&-\int_{t-h(t)}^{t}\dot{x}^\top(s)S_4\dot{x}(s)\mathrm{d}s-2[x^\top N_1+x^\top(t-h(t))N_2]\int_{t-h(t)}^{t}\dot{x}(s)\mathrm{d}s\\ &=-\int_{t-h(t)}^{t}([x^\top(s)N+\dot{x}^\top(s)S_4]S_4^{-1}[N^\top x(s)+S_4\dot{x}(s)])\mathrm{d}s+\int_{t-h(t)}^{t}(x^\top(s)NS_4^{-1}N^\top x(s))\mathrm{d}s,\end{aligned}$$

where $N^\top=[N_1^\top \quad N_2^\top]^\top$.

Because $S_4$ is positive, then the first term on the right-hand side must be negative, thus by introducing $p(t)$ and $\bar{\varOmega}(t)$, (4.7) can be rewritten as

$$\dot{V}(t)=p^\top(t)\bar{\varOmega}(t)p(t)-[x^\top(t)Qx(t)+u^\top(t)Ru(t)],\tag{4.8}$$

where

$$\bar{\varOmega}(t)=\begin{bmatrix}\bar{\varOmega}_1 & \bar{\varOmega}_2 & \bar{\varOmega}_3\\ \bar{\varOmega}_2^\top & \bar{\varOmega}_4 & \bar{\varOmega}_5\\ \bar{\varOmega}_3^\top & \bar{\varOmega}_5^\top & \bar{\varOmega}_6\end{bmatrix},\quad p(t)=\begin{bmatrix}x(t)\\ x(t-h(t))\\ \xi(t)\end{bmatrix},$$

$$\begin{aligned}\bar{\varOmega}_1 &= S_1(A+\varDelta A(t)+(B+\varDelta B(t))K)+(A+\varDelta A(t)+(B+\varDelta B(t))K)^\top S_1+S_3\\ &\quad+h(A+\varDelta A(t)+(B+\varDelta B(t))K)^\top S_4(A+\varDelta A(t)+(B+\varDelta B(t))K)+N_1^\top+N_1+Q\\ &\quad+K^\top RK+hN_1S_4^{-1}N_1^\top,\\ \bar{\varOmega}_2 &= S_1(A_d+\varDelta A_d(t))+h(A+\varDelta A(t)+(B+\varDelta B(t))K)^\top S_4(A_d+\varDelta A_d(t))+N_2^\top-N_1,\\ \bar{\varOmega}_3 &= S_1(B+\varDelta B(t))KP-(\varDelta A(t)+\varDelta B(t)K)^\top T^\top S_2+h(A+\varDelta A(t)\\ &\quad+(B+\varDelta B(t))K)^\top S_4(B+\varDelta B(t))KP+K^\top RKP,\\ \bar{\varOmega}_4 &= -(1-\dot{h}(t))S_3+h(A_d+\varDelta A_d)^\top S_4(A_d+\varDelta A_d)-N_2-N_2^\top+hN_2S_4^{-1}N_2,\\ \bar{\varOmega}_5 &= -(A_d+\varDelta A_d(t))^\top T^\top S_2+h(A_d+\varDelta A_d)^\top S_4(B+\varDelta B(t))KP,\\ \bar{\varOmega}_6 &= S_2(D-T\varDelta B(t)KP)+(D-T\varDelta B(t)KP)^\top S_2+h((B+\varDelta B(t))KP)^\top S_4(B+\varDelta B(t))KP\\ &\quad+P^\top K^\top RKP.\end{aligned}$$

Under the condition $\bar{\Omega}(t) < 0$, it is equivalent to the LMI below:

$$\begin{bmatrix} \bar{\Omega}_7 & \bar{\Omega}_8 & \bar{\Omega}_9 & \bar{\Omega}_{13} \\ \bar{\Omega}_8^\top & \bar{\Omega}_{10} & \bar{\Omega}_{11} & \bar{\Omega}_{14} \\ \bar{\Omega}_9^\top & \bar{\Omega}_{11}^\top & \bar{\Omega}_{12} & \bar{\Omega}_{15} \\ \bar{\Omega}_{13}^\top & \bar{\Omega}_{14}^\top & \bar{\Omega}_{15}^\top & -(hS_4)^{-1} \end{bmatrix} + hNS_4^{-1}N^\top < 0, \tag{4.9}$$

where

$$\begin{aligned}
\bar{\Omega}_7 &= S_1(A + \varDelta A(t) + (B + \varDelta B(t))K) + (A + \varDelta A(t) + (B + \varDelta B(t))K)^\top S_1 + S_3 + N_1^\top \\
&\quad + N_1 + Q + K^\top RK, \\
\bar{\Omega}_8 &= S_1(A_d + \varDelta A_d(t)) + N_2^\top - N_1, \\
\bar{\Omega}_9 &= S_1(B + \varDelta B(t))KP - (\varDelta A(t) + \varDelta B(t)K)^\top T^\top S_2 + K^\top RKP, \\
\bar{\Omega}_{10} &= -(1 - \dot{h}(t))S_3 - N_2 - N_2^\top, \quad \bar{\Omega}_{11} = -(A_d + \varDelta A_d(t))^\top T^\top S_2, \\
\bar{\Omega}_{12} &= S_2(D - T\varDelta B(t)KP) + (D - T\varDelta B(t)KP)^\top S_2 + P^\top K^\top RKP, \\
\bar{\Omega}_{13} &= (A + \varDelta A(t) + (B + \varDelta B(t))K)^\top, \quad \bar{\Omega}_{14} = (A_d + \varDelta A_d(t))^\top, \\
\bar{\Omega}_{15} &= P^\top K^\top (B + \varDelta B(t))^\top.
\end{aligned}$$

Further, we can decompose and rebuild (4.9) as follows:

$$\begin{bmatrix} \begin{bmatrix} \varXi_1 & \varXi_2 & 0 \\ \varXi_2^\top & \varXi_3 & \varXi_4 \\ 0 & \varXi_4^\top & \varXi_5 \end{bmatrix} & \begin{matrix} \varPi_1 & N_1 & N_2^\top - N_1 \\ \varPi_2 & N_2 & 0 \\ \varPi_3 & 0 & 0 \end{matrix} \\ \begin{matrix} \varPi_1^\top & \varPi_2^\top & \varPi_3^\top \\ N_1^\top & N_2^\top & 0 \\ N_2 - N_1^\top & 0 & 0 \end{matrix} & \begin{matrix} -(hS_4)^{-1} & 0 & 0 \\ 0 & -h^{-1}S_4 & 0 \\ 0 & 0 & -I \end{matrix} \end{bmatrix} < 0, \tag{4.10}$$

where

$$\begin{aligned}
\varXi_1 &= S_1(A + \varDelta A(t) + (B + \varDelta B(t))K) + (A + \varDelta A(t) + (B + \varDelta B(t))K)^\top S_1 + S_3 + N_1^\top \\
&\quad + N_1 + Q + K^\top RK, \\
\varXi_2 &= S_1(A_d + \varDelta A_d(t)), \quad \varXi_3 = -(1 - \dot{h}(t))S_3 - N_2 - N_2^\top, \quad \varXi_4 = A_d^\top T^\top S_2, \\
\varXi_5 &= S_2(D - T\varDelta B(t)KP) + (D - T\varDelta B(t)KP)^\top S_2 + P^\top K^\top RKP, \\
\varPi_1 &= (A + \varDelta A(t) + (B + \varDelta B(t))K)^\top, \quad \varPi_2 = (A_d + \varDelta A_d(t))^\top, \\
\varPi_3 &= P^\top K^\top (B + \varDelta B(t))^\top.
\end{aligned}$$

Applying Fact 2.2 for uncertainties in (4.10), pre- and postmultiplying with $\mathrm{diag}(S_1^{-1}, I, I, I)$ on both sides of the upper left side decomposed matrix

$$\varXi = \begin{bmatrix} \varXi_1 & \varXi_2 & 0 \\ \varXi_2^\top & \varXi_3 & \varXi_4 \\ 0 & \varXi_4^\top & \varXi_5 \end{bmatrix}$$

to get

$$\bar{\Xi} = \begin{bmatrix} \bar{\Xi}_1 & \bar{\Xi}_2 & 0 \\ \bar{\Xi}_2^\top & \bar{\Xi}_3 & \bar{\Xi}_4 \\ 0 & \bar{\Xi}_4^\top & \bar{\Xi}_5 \end{bmatrix},$$

where

$$\bar{\Xi}_1 = (A + \Delta A(t))X + X(A + \Delta A(t))^\top + X - BR^{-1}B^\top + X^\top(N_1^\top + N_1 + Q)X,$$
$$\bar{\Xi}_2 = (A_d + \Delta A_d(t)), \quad \bar{\Xi}_3 = -(1 - \dot{h})S_3 - N_2 - N_2^\top, \quad \bar{\Xi}_4 = \Xi_4, \quad \bar{\Xi}_5 = \Xi_5$$

and using Schur complement, (4.10) leads to the new LMI (4.1). □

## 5. A full-order observer-based guaranteed cost approach

By following Ishitobi & Miyachi (2008), the result of the full-order observer-based guaranteed cost approach for uncertain time-varying delay is introduced by Theorem 5.1 and the feedback gain is represented as

$$K = -R^{-1}B^\top P_1, \tag{5.1}$$

where $P_1$ is a symmetric positive-definite matrix.

THEOREM 5.1 Under Assumption 2.1, if the following matrix inequalities optimization problem: $\min\{\mathrm{tr}(M_1) + tr(M_2) + \mathrm{tr}(M_3) + \mathrm{tr}(M_4)\}$ subject to

$$\begin{bmatrix}
\Psi_1 & X^\top & XE_A^\top & A_d & G_1^\top & 0 & 0 & 0 & 0 & 0 & 0 \\
X & -(M_x + Q^{-1}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
E_A X & 0 & -\alpha_{(1,5)} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
A_d^\top & 0 & 0 & \Psi_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
G_1 & 0 & 0 & 0 & \Psi_3 & P_1 B & G_2^\top & R_1 D_A & R_1 D_B & R_1 D_{Ad} & R_1 A_d \\
0 & 0 & 0 & 0 & B^\top P_1 & -R & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & G_2 & 0 & -\alpha_{(4,8)} I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & D_A^\top R_1 & 0 & 0 & -\alpha_5^{-1} I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & D_B^\top R_1 & 0 & 0 & 0 & -\alpha_{(6,8)}^{-1} I & 0 & 0 \\
0 & 0 & 0 & 0 & D_{Ad}^\top R_1 & 0 & 0 & 0 & 0 & -\alpha_7^{-1} I & 0 \\
0 & 0 & 0 & 0 & A_d^\top R_1 & 0 & 0 & 0 & 0 & 0 & -(1-d)R_2
\end{bmatrix} < 0, \tag{5.2}$$

$$\begin{bmatrix} -M_1 & G_x \\ G_x^\top & -P_1^{-1} \end{bmatrix} < 0, \tag{5.3}$$

$$\begin{bmatrix} -M_2 & H_x \\ H_x^\top & -R_1^{-1} \end{bmatrix} < 0, \tag{5.4}$$

$$\begin{bmatrix} -M_3 & E_x \\ E_x^\top & -P_2^{-1} \end{bmatrix} < 0, \tag{5.5}$$

$$\begin{bmatrix} -M_4 & F_x \\ F_x^\top & -R_2^{-1} \end{bmatrix} < 0, \tag{5.6}$$

where

$$\Psi_1 = AX + XA^\top - BR^{-1}B^\top + \alpha_1 D_A D_A^\top + (\alpha_2 + \alpha_4) D_B D_B^\top$$
$$+ (\alpha_2^{-1} + \alpha_6^{-1}) BR^{-1} E_B^\top E_B R^{-1} B^\top + \alpha_3 D_{Ad} D_{Ad}^\top,$$
$$\Psi_2 = -(1-d)P_2 + (\alpha_3^{-1} + \alpha_7^{-1}) E_{Ad} E_{Ad}^\top, \quad \Psi_3 = R_1 A + A^\top R_1 - C^\top \hat{L} - \hat{L} C + R_2$$
$$\alpha_{(1,5)} = (\alpha_1 + \alpha_5), \quad \alpha_{(4,8)} = (\alpha_4 + \alpha_8), \quad \alpha_{(6,8)}^{-1} = (\alpha_6^{-1} + \alpha_8^{-1}),$$
$$G_1 = \sqrt{2} P_1 B R^{-1} B^\top, \quad G_2 = E_B R^{-1} B^\top P_1,$$
$$G_x = hI_n, \quad H_x = hI_n, \quad E_x = hI_n, \quad F_x = hI_n,$$

has a set of solutions $P_1 > 0$, $R_1 > 0$, $P_2 > 0$, $R_2 > 0$, $M_x > 0$, $X > 0$, $\alpha_2 > 0$, $\bar{\alpha}_2 > 0$, $\alpha_3 > 0$, $\bar{\alpha}_3 > 0$, $\alpha_5 > 0$, $\bar{\alpha}_5 > 0$, $\alpha_8 > 0$ and $\bar{\alpha}_8 > 0$, which satisfy the relation $\alpha_2^{-1} = \bar{\alpha}_2$, $\alpha_3^{-1} = \bar{\alpha}_3$, $\alpha_5^{-1} = \bar{\alpha}_5$ and $\alpha_8^{-1} = \bar{\alpha}_8$, then the full-order observer-based control law (5.8–5.9), (2.8) is a guaranteed cost controller which gives the minimum expected value of the guaranteed cost

$$E[J^*] = E\left[ q^\top(0) \begin{bmatrix} P_1 & 0 \\ 0 & R_1 \end{bmatrix} q(0) + \int_{-h(0)}^{0} q^\top(s) \begin{bmatrix} P_2 & 0 \\ 0 & R_2 \end{bmatrix} q(s) \mathrm{d}s \right]$$
$$= \min\{\mathrm{tr}(M_1) + \mathrm{tr}(M_2) + \mathrm{tr}(M_3) + tr(M_4)\}, \tag{5.7}$$

where $q(\cdot) = [x(\cdot)^\top \quad \mathrm{e}(\cdot)^\top]^\top$.

*Proof.* Similar to that of minimal-order observer case, we obtain Theorem 5.1. The designed full-order observer is formed by

$$\dot{\hat{x}}(t) = A\hat{x}(t) + A_d \hat{x}(t - h(t)) + Bu(t) + L(y(t) - \hat{y}(t)), \tag{5.8}$$
$$\hat{y}(t) = C\hat{x}(t). \tag{5.9}$$

Define error and time derivative of error, respectively, as

$$e(t) = x(t) - \hat{x}(t), \tag{5.10}$$
$$\dot{e}(t) = \dot{x}(t) - \dot{\hat{x}}(t). \tag{5.11}$$

Equations (2.1–2.3) and (5.8–5.11) result in the closed-loop system

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} \Phi_1 & \Phi_2 & \Phi_3 & 0 \\ \Phi_4 & \Phi_5 & \Phi_6 & \Phi_7 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \\ e(t) \\ e(t-h(t)) \end{bmatrix}, \tag{5.12}$$

where

$$\Phi_1 = A + \Delta A(t) + (B + \Delta B(t))K, \quad \Phi_2 = A_d + \Delta A_d(t), \quad \Phi_3 = (B + \Delta B(t))K,$$
$$\Phi_4 = \Delta A(t) + \Delta B(t)K, \quad \Phi_5 = \Delta A_d(t), \quad \Phi_6 = A - \Delta B(t)K - LC, \quad \Phi_7 = A_d.$$

Consider the Lyapunov function

$$V = q^\top(t)\begin{bmatrix} P_1 & 0 \\ 0 & R_1 \end{bmatrix} q(t) + \int_{-h(t)}^{\top} q^\top(t)\begin{bmatrix} P_2 & 0 \\ 0 & R_2 \end{bmatrix} q(t)\mathrm{d}t. \quad (5.13)$$

Using the time derivative of (5.13) along with (5.12) and introducing $\bar{z}_f(t)$ and $\Omega_f(t)$ result in

$$\dot{V}(t) = \bar{z}_f^\top(t)\Omega_f(t)\bar{z}_f(t) - (x^\top(t)Qx(t) + u^\top(t)Ru(t)), \quad (5.14)$$

where

$$\bar{z}_f(t) = \begin{bmatrix} x(t) \\ x(t-h(t)) \\ e(t) \\ e(t-h(t)) \end{bmatrix}, \quad \Omega_f(t) = \begin{bmatrix} \Omega_{f1} & \Omega_{f2} & \Omega_{f3} & 0 \\ \Omega_{f2}^\top & \Omega_{f4} & \Omega_{f5} & 0 \\ \Omega_{f3}^\top & \Omega_{f5}^\top & \Omega_{f6} & \Omega_{f7} \\ 0 & 0 & \Omega_{f7}^\top & \Omega_{f8} \end{bmatrix},$$

$$\Omega_{f1} = P_1(A + \varDelta A(t) + (B + \varDelta B(t))K) + (A + \varDelta A(t) + (B + \varDelta B(t))K)^\top P_1 + P_2 + Q + K^\top RK,$$

$$\Omega_{f2} = P_1(A_d + \varDelta A_d(t)), \quad \Omega_{f3} = -P_1(B + \varDelta B(t))K + (\varDelta A(t) + \varDelta B(t)K)^\top R_1 + K^\top RK,$$

$$\Omega_{f4} = -(1-\dot{h}(t))P_2, \quad \Omega_{f5} = \varDelta A_d^\top R_1,$$

$$\Omega_{f6} = (A - \varDelta B(t)K - LC)^\top R_1 + R_1(A - \varDelta B(t)K - LC) + R_2 + K^\top RK,$$

$$\Omega_{f7} = R_1 A_d, \quad \Omega_{f8} = -(1-\dot{h}(t))R_2.$$

Under similar condition with (3.12), (5.14) leads to (3.13) for any $x(t) \neq 0$ and the closed-loop system is also asymptotically stable. Further, applying Fact (2.2), pre- and postmultiplying with $\mathrm{diag}(P_1^{-1}, I, I, I)$ and letting $P_1^{-1} = X, \hat{L} = R_1 L$ yield

$$\Omega_f(t) = \begin{bmatrix} \Omega_{f9} & \Omega_{f10} & \Omega_{f11} & 0 \\ \Omega_{f10}^\top & \Omega_{f12} & 0 & 0 \\ \Omega_{f11}^\top & 0 & \Omega_{f13} & \Omega_{f14} \\ 0 & 0 & \Omega_{f14}^\top & \Omega_{f15} \end{bmatrix} < 0, \quad (5.15)$$

$$\begin{aligned}\Omega_{f9} = {} & XA^\top + AX - BR^{-1}B^\top + X^\top(P_2 + Q)X + \alpha_1 D_A D_A^\top + (\alpha_1^{-1} + \alpha_5^{-1})XE_A^\top E_A X \\ & + (\alpha_2^{-1} + \alpha_6^{-1})BR^{-1}E_B^\top E_B R^{-1}B^\top + (\alpha_2 + \alpha_4)D_B D_B^\top + \alpha_3 D_{Ad} D_{Ad}^\top,\end{aligned}$$

$$\Omega_{f10} = A_d, \quad \Omega_{f11} = 2BR^{-1}B^{-1}P_1, \quad \Omega_{f12} = -(1-\dot{h}(t))P_2 + (\alpha_3^{-1} + \alpha_7^{-1})E_{Ad}^\top E_{Ad},$$

$$\begin{aligned}\Omega_{f13} = {} & A^\top R_1 + R_1 A - C^\top \hat{L}^\top - \hat{L}C + R_2 + P_1 BR^{-1}B^\top P_1 + (\alpha_4^{-1} + \alpha_8^{-1})P_1 BR^{-1}E_B^\top E_B R^{-1}B^\top P_1 \\ & + \alpha_5 R_1 D_A D_A^\top R_1 + \alpha_7 R_1 D_{Ad} D_{Ad}^\top R_1 + (\alpha_6 + \alpha_8)R_1 D_B D_B^\top R_1,\end{aligned}$$

$$\Omega_{f14} = R_1 A_d, \quad \Omega_{f15} = -(1-\dot{h}(t))R_2.$$

By using Schur complement, (5.15) is equivalent to (5.2).

Next, the optimal expected guaranteed cost is calculated as

$$\begin{aligned}E[J^*] = {} & \mathrm{tr}P_1 E[x(0)x^\top(0)] + \mathrm{tr}R_1 E[e(0)e^\top(0)] + \mathrm{tr}P_2 E\left[\int_{-h(0)}^{0} x(s)x^\top(s)\mathrm{d}s\right] \\ & + \mathrm{tr}R_2 E\left[\int_{-h(0)}^{0} e(s)\mathrm{e}^\top(s)\mathrm{d}s\right]. \quad (5.16)\end{aligned}$$

From (5.16), one can easily see that

$$\mathrm{tr}P_1E[x(0)x^\top(0)] = \mathrm{tr}P_1G_xG_x^\top < \mathrm{tr}(M_1), \tag{5.17}$$

$$\mathrm{tr}R_1E[e(0)\mathrm{e}^\top(0)] = \mathrm{tr}R_1H_xH_x^\top < \mathrm{tr}(M_2), \tag{5.18}$$

$$\mathrm{tr}P_2E\left[\int_{-h(0)}^{0} x^\top(s)x(s)\mathrm{d}s\right] = \mathrm{tr}P_2E_xE_x^\top < \mathrm{tr}(M_3), \tag{5.19}$$

$$\mathrm{tr}R_2E\left[\int_{-h(0)}^{0} \mathrm{e}^\top(s)e(s)\mathrm{d}s\right] = \mathrm{tr}R_2F_xF_x^\top < \mathrm{tr}(M_4). \tag{5.20}$$

Equations (5.17–5.20) are equivalent to (5.3–5.6). □

## 6. An illustrative example

Consider a system (2.1–2.3) with

$$A = \begin{bmatrix} -3 & 0 & -2 & 0 \\ 0 & -2 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 2 \\ -6 \\ 1 \end{bmatrix}, \quad C = [\mathrm{O}_2 \quad I_2], \quad m(0) = 0_4,$$

$$\Sigma(0) = I_4, \quad R = 9, \quad Q = \mathrm{diag}(7, 15, 1, 3), \quad D_A = \begin{bmatrix} 0.1I_2 & \mathrm{O}_2 \\ \mathrm{O}_2 & \mathrm{O}_2 \end{bmatrix}, \quad E_A = \begin{bmatrix} 0.3I_2 & 0.3I_2 \\ \mathrm{O}_2 & \mathrm{O}_2 \end{bmatrix},$$

$$D_B = \mathrm{diag}(0.3, 0.1, 0.3, 0.1), \quad E_B = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^\top,$$

$$A_d = 0.05A, \quad D_{Ad} = 0.01D_A, \quad E_{Ad} = 0.01E_A, \quad h(0) = h = 0.5, \quad d = 0.35.$$

- *Case*$(I)$
  Applying the minimal-order observer-based approach with $k_{\max}$=1000, $\gamma_{\min}$=0.0001, $\kappa$=0.0001 and initial $\gamma = 100$, we obtain a solution

$$L = \begin{bmatrix} -0.1019 & -0.0957 \\ -0.0783 & -0.1401 \end{bmatrix}, \quad K = [-0.4018 \quad -0.4517 \quad 0.5557 \quad -0.5088],$$

$$E_m = \begin{bmatrix} -2.3234 & -0.3102 \\ -0.1756 & -1.3073 \end{bmatrix}, \quad W = \begin{bmatrix} 0.1019 & 0.0957 \\ 0.0783 & 0.1401 \\ 1.0000 & 0 \\ 0 & 1.0000 \end{bmatrix}, \quad \bar{\gamma} = E[J^*] = 18.9276.$$

In the iterations 1–158 of Fig. 1, $\bar{\gamma}$ does not vary and be kept to the initial $\gamma$ because the feasible solution is searched. After that, this algorithm optimizes the guaranteed cost candidate. Figures 2 and 3 depict trajectories of states and estimate errors with $x(0) = [-1, 2, 3, -2]^\top$, $x(-h(0)) = [1, -2, -3, 2]^\top$ and $F_A(t) = F_{Ad}(t) = F_B(t) = I$.
*Case* $(Ia)$. For the relaxed constraint of time derivative of time-varying delay, two cases $d < 1$ and $d > 1$ are provided. Replacing LMI in (3.2) by (4.1), a controller gain, an observer gain and an expected value of guaranteed cost, respectively, are

$$K = [0.0002 \quad -0.0018 \quad 0.0065 \quad -0.0021], L = \begin{bmatrix} -0.0458 & 0.0120 \\ 0.0116 & -0.2345 \end{bmatrix}, \bar{\gamma} = E[J^*] = 0.0681.$$

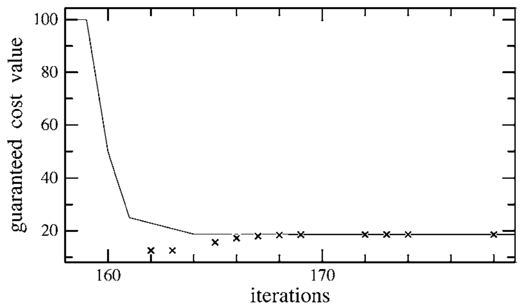



FIG. 1. Trajectories of the guaranteed cost. The marks × show that a feasible solution cannot be obtained for the guaranteed cost candidate $\bar{\gamma}$ and the optimal performance index is greater than ×.

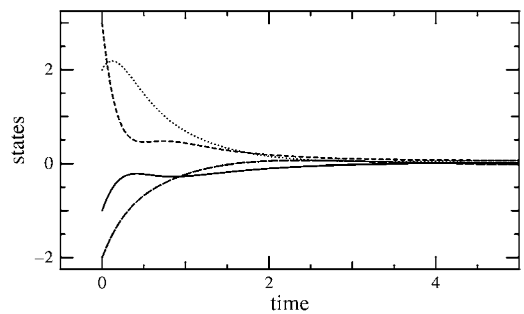



FIG. 2. Trajectories of states $x_1$(—), $x_2$($\cdots$), $x_3$(- -) and $x_4$(– –).

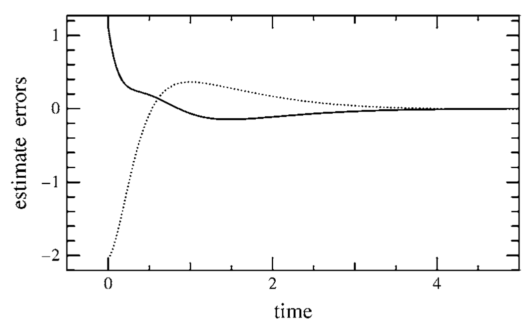



FIG. 3. Trajectories of estimate errors $\xi_1$(—) and $\xi_2$(- -).

For $h = 1.5$ and $d = 1.1$, we obtain

$$K = [-0.0009 \quad -0.0013 \quad 0.0037 \quad -0.0010], \quad L = \begin{bmatrix} -0.1623 & 0.0017 \\ 0.0023 & -0.2362 \end{bmatrix}, \bar{\gamma} = E[J^*] = 0.0658.$$

- *Case (II)*
  By the full-order observer-based approach, the obtained observer and controller gain for uncertain time-varying delay system (2.1–2.3) are

$$L_f = 10^{24} \times \begin{bmatrix} -0.5528 & 0.1757 \\ 0.1187 & -0.4753 \\ 1.4625 & -0.4577 \\ -0.4577 & 1.6177 \end{bmatrix}, \quad K_f = [-0.0218 \quad -0.0080 \quad 2.8922 \quad -1.3604].$$

Note that the observer gain is extremely large so that a practical difficulty is remained.

## 7. Conclusions

This paper discusses a guaranteed cost controller design with a minimal-order observer for uncertain time-varying delay systems. A sufficient condition for the existence of state feedback guaranteed cost controllers is derived on the basis of the LMI feasible solutions. The optimal cost control is provided by minimizing the upper bound of the guaranteed cost control. To show the advantage of a minimal-order observer-based guaranteed cost controller, the problem of a full-order observer-based case is also treated as comparison. A numerical example is given to illustrate the proposed method.

## References

BOYD, S. P., EL GHAOUI, L., FERON, E. & BALAKRISHNAN, V. (1994) *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA: SIAM.

CHANG, S. S. L. & PENG, T. K. C. (1972) Adaptive guaranteed cost control of systems with uncertain parameters. *IEEE Trans. Autom. Control*, **17**, 474–483.

CHEN, W.-H., GUAN, Z.-H. & LU, X. (2004) Delay-dependent output feedback guaranteed cost control for uncertain time-delay systems. *Automatica*, **40**, 1263–1268.

EL GHAOUI, L., OUSTRY, F. & AITRAMI, M. (1997) A cone complementarity linearization algorithm for static output-feedback and related problems. *IEEE Trans. Autom. Control*, **42**, 1171–1176.

HE, Y., WANG, Q.-G., LIN, C. & WU, M. (2007) Delay-range-dependent stability for systems with time-varying delay. *Automatica*, **43**, 371–376.

ISHITOBI, M. & MIYACHI, M. (2008) An ILMI approach to guaranteed cost controllers with an observer. *Dyn. Contin. Discr. Impuls. Syst. B*, **15**, 317–326.

LEIBFRITZ, F. (2001) An LMI-based algorithm for designing suboptimal static $H_2/H_\infty$ output feedback controllers. *SIAM J. Control Optim.*, **39**, 1711–1735.

LIEN, C. H. (2005) Guaranteed cost observer-based controls for a class of uncertain neutral time-delay systems. *J. Optim. Theory Appl.*, **126**, 137–156.

MAHMOUD, M. S. (2001) Control of uncertain state-delay systems: guaranteed cost approach. *IMA J. Math. Control Inf.*, **18**, 109–128.

MAHMOUD, M. S. & ZRIBI, M. (2003) Guaranteed cost observer-based control of uncertain time-lag systems. *Comput. Electr. Eng.*, **29**, 193–212.

YU, K. W. & LIEN, C. H. (2007) Delay-dependent conditions for guaranteed cost observer-based control of uncertain neutral systems with time-varying delays *IMA J. Math. Control Inf.*, **24**, 383–394.

YU, L. & CHU, J. (1999) An LMI approach to guaranteed cost control of linear uncertain time-delay systems. *Automatica*, **35**, 1155–1159.

# Table of Contents

*Volume 29 Issue 1 March 2012*

For checked items
view abstracts download to citation manager Go Clear

## Articles

Emilia Fridman

Stability of singularly perturbed functional-differential systems: spectrum analysis and LMI approaches

*IMA J Math Control Info (2012) 29 (1): 79-111 doi:10.1093/imamci/dnr027*

Abstract
Full Text (PDF)
Permissions
Permissions

☐ Select this article

Erwin Susanto,
Mitsuaki Ishitobi,
Sadaaki Kunimatsu,
and Daisuke Matsunaga

A minimal-order observer-based guaranteed cost controller for uncertain time-varying delay systems

*IMA J Math Control Info (2012) 29 (1): 113-132 doi:10.1093/imamci/dnr029*

Abstract
Full Text (PDF)
Permissions
Permissions

☐ Select this article

Xiaodi Li and
Xilin Fu

Lag synchronization of chaotic delayed neural networks via impulsive control

*IMA J Math Control Info (2012) 29 (1): 133-145 doi:10.1093/imamci/dnr030*

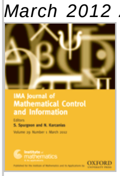
Abstract
Full Text (PDF)
Permissions
Permissions

For checked items
○ view abstracts ○ download to citation manager [Go] [Clear]



**This Issue**

*March 2012 29 (1)*





Find articles in this issue containing these words:

[Search This Issue] [GO]

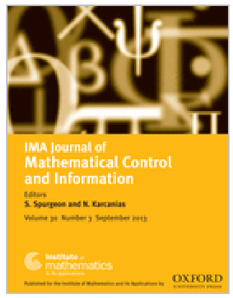
Advance Access

Search this journal:

Advanced »

## Current Issue

1. September 2013 30 (3)

1. 


1. Alert me to new issues

## The Journal

- About this journal
- Rights & Permissions
- Dispatch date of the next issue
- Conflict of Interest Policy
- We are mobile – find out more
- This journal is a member of the Committee on Publication Ethics (COPE)



**Published on behalf of**

The Institute of Mathematics and its Applications

**Impact Factor: 0.741**
**5-Yr impact factor: 0.596**

**Most**

Most Read
Most Cited
Most Read
Most Cited

**Most Read**

1. Planning and control of three-dimensional multi-agent formations
2. A Lyapunov-based design of a modified super-twisting algorithm for the Heisenberg system
3. Stabilization of quasi-one-sided Lipschitz nonlinear systems
4. Stability analysis for discrete-time switched linear singular systems: average dwell time approach
5. An algebraic framework for information theory: classical information

» View all Most Read articles

**Most Cited**

1. Parametric Identification of Transient Signals
2. A Maximum Principle for Nonconservative Self-Adjoint Systems
3. Applications of a Maximum Principle for the Structural Control of Laminated Composite Plates
4. A New Approach to the H{infty} Design of Optimal Digital Linear Filters
5. Optimal control of a hyperbolic system with time-varying lags

» View all Most Cited articles

Online ISSN 1471-6887 - Print ISSN 0265-0754