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A Minimal Order Observer-based Guaranteed Cost Controller for Uncertain Time-delay Systems

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Abstract: This paper considers a design scheme of a minimal order observer-based guaranteed cost controller for linear uncertain time-delay systems. A sufficient condition of the asymptotic stabilization is guaranteed via linear matrix inequality (LMI) feasible solutions. A numerical example is given to illustrate the proposed method.

Keywords: guaranteed cost control, time-delay systems, a minimal order observer, LMI

1. INTRODUCTION

In many practical control systems, time-delay phenomena may cause instability and bad performance on the controlled system [1]. Hence, much effort has been devoted to the robust stability and stabilization of the feedback control system with time delay. Moreover, it is desirable to design a controller which not only achieves the stability of the uncertain system but also guarantees an adequate level of performance. One of the approaches to solve this problem is guaranteed cost control method [2].

In many cases, full states of the system can not be measured directly because of some reasons such as poor plant knowlegde, costing problems, etc. Hence, an observer-based control may be more suitable than a state feedback control in such situation. Mahmoud and Zribi [3] developed the guaranteed cost observer-based controllers for uncertain time-lag systems via solutions of linear matrix inequalities though the cost function is not optimized. Recently, Lien [1] considered observer-based guaranteed cost control for uncertain neutral systems with known and unknown time delay. However, the convex optimization problem contains the equality constraint. More recently, Ishitobi and Miyachi investigated an observer-based guaranteed cost controller [4], but the obtained observer gain is extremely large and some of the closed loop poles are located far in the left half plane.

All of the previous studies considered a full order observer-based guaranteed cost control. To the best of our knowledge, no results on the observer-based guaranteed cost control with a minimal order observer are available in the past. This motivates us to concern a problem on minimal order observer-based guaranteed cost control.

This paper proposes a design method of a minimal order observer-based guaranteed cost controller for time-delay systems via an LMI technique.

2. PROBLEM STATEMENT

Consider a continuous time uncertain system with time-delay in the form

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= (A+\Delta A(t))\boldsymbol{x}(t)\\ &\quad +(A_d+\Delta A_d(t))\boldsymbol{x}(t-h)+B\boldsymbol{u}(t) \qquad (1)\end{aligned}$$

$$\boldsymbol{y}(t) = C\boldsymbol{x}(t) \qquad (2)$$

$$\boldsymbol{x}(t) = \boldsymbol{\psi}(t), t\in[-h,0] \qquad (3)$$

where $h>0$ is the given constant time-delay factor in the states, $\boldsymbol{x}(t)\in\Re^n$ is the state vector, $\boldsymbol{u}(t)\in\Re^r$ is the control input vector, $\boldsymbol{y}(t)\in\Re^m$ is the measured output vector, $\boldsymbol{x}(t)=\boldsymbol{\psi}(t)$ is a given continuous time vector valued initial function of $t\in[-h,0]$, matrices A, A_d and B are known constant real-valued matrices with appropriate dimensions, and C is restricted to the form of $C=[O\ I_m]$. Matrices $\Delta A(t)$ and $\Delta A_d(t)$ denote real-valued matrix functions representing parameter uncertainties.

We assume that the parameter uncertainties $\Delta A(t)$ and $\Delta A_d(t)$ satisfy the following relations

$$\Delta A(t) = D_A F_A(t) E_A, \Delta A_d(t) = D_{Ad} F_{Ad}(t) E_{Ad}, \qquad (4)$$

where $F_A(t)$ and $F_{Ad}(t)$ are unknown time-varying and deterministic matrices satisfying

$$F_A^T(t)F_A(t)\le I, F_{Ad}^T(t)F_{Ad}(t)\le I, \qquad (5)$$

and D_A, D_{Ad}, E_A and E_{Ad} are constant real-valued known matrices with appropriate dimensions.

It is also assumed that the initial state variable $\boldsymbol{x}(0)$ is unknown, but their mean and covariance are known, respectively as

$$E\left[\boldsymbol{x}(0)\right] = \boldsymbol{m}_0 \qquad (6)$$

$$E\left[(\boldsymbol{x}(0)-\boldsymbol{m}_0)(\boldsymbol{x}(0)-\boldsymbol{m}_0)^T\right] = \Sigma_0 > O \qquad (7)$$

where $E\left[\cdot\right]$ denotes the expected value operator.

The problem considered here is to design a minimal order observer

$$\dot{\boldsymbol{z}}(t) = D\boldsymbol{z}(t)+E\boldsymbol{y}(t)+F\boldsymbol{u}(t) \qquad (8)$$

$$\hat{\boldsymbol{x}}(t) = P\boldsymbol{z}(t)+W\boldsymbol{y}(t) \qquad (9)$$

and a controller

$$\boldsymbol{u}(t) = K\hat{\boldsymbol{x}}(t) \qquad (10)$$

with

$$D = A_{11}+LA_{21},\ PT+WC = I_n,$$

$$F = TB, TA-DT = EC,\ A = \left[\begin{array}{c|c} A_{11} & A_{12}\\ \hline A_{21} & A_{22}\end{array}\right],$$

$$P = \left[I_{n-m}\ \boldsymbol{0}\right]^T,\ T = \left[I_{n-m}\ L\right]$$

so as to achieve an upper bound on the following quadratic performance index

$$E[J] = E\left[\int_0^{\infty} (\boldsymbol{x}^T(t)Q\boldsymbol{x}(t) + \boldsymbol{u}^T(t)R\boldsymbol{u}(t))dt\right] \tag{11}$$

associated with the uncertain time-delay system (1)-(3), where Q and R are given symmetric positive-definite matrices.

3. MAIN RESULTS

The main result of this study is given by Theorem 1. Here, it is assumed that the feedback gain matrix is given to the form

$$K = -R^{-1}B^TS_1 \tag{12}$$

where S_1 is a symmetric positive-definite matrix.

Theorem 1. If the following matrix inequalities optimization problem; $\min\{\gamma_0+\gamma_1+\gamma_2+\gamma_3+\gamma_4+tr(M_1)+tr(M_2)\}$ subject to

$$\begin{bmatrix} \Lambda_0 & X^T & X^T & G_5^T & G_5^T & A_d & 0 & 0 & 0 & 0 & 0 \\ X & -M_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X & 0 & -Q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ G_5 & 0 & 0 & -\zeta I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ G_5 & 0 & 0 & 0 & -\omega I & 0 & 0 & 0 & 0 & 0 & 0 \\ A_d^T & 0 & 0 & 0 & 0 & \Lambda_1 & G_4^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & G_4 & \Lambda_2 & G_1^T & G_2^T & G_3^T & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & G_1 & -R & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & G_2 & 0 & -\bar{\omega}I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & G_3 & 0 & 0 & -\bar{\lambda}I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_3 \end{bmatrix} < 0 \tag{13}$$

$$\sum_{k=1}^{n} \boldsymbol{e}_{nk}^T\Theta_0\boldsymbol{e}_{nk} < \gamma_0, \ \sum_{k=1}^{m} \boldsymbol{e}_{mk}^T\Theta_1\boldsymbol{e}_{mk} < \gamma_1$$
$$\sum_{k=1}^{m} \boldsymbol{e}_{mk}^T\Theta_2\boldsymbol{e}_{mk} < \gamma_2, \ \sum_{k=1}^{m} \boldsymbol{e}_{mk}^T\Theta_3\boldsymbol{e}_{mk} < \gamma_3 \tag{14}$$

$$\begin{bmatrix} -\gamma_4 & \boldsymbol{v}_1^TY^T & \boldsymbol{v}_2^TY^T & \cdots & \boldsymbol{v}_m^TY^T \\ Y\boldsymbol{v}_1 & -S_2 & & & \vdots \\ Y\boldsymbol{v}_2 & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ Y\boldsymbol{v}_m & \cdots & \cdots & \cdots & -S_2 \end{bmatrix} < 0 \tag{15}$$

$$\begin{bmatrix} -M_1 & E_x \\ E_x^T & -S_3^{-1} \end{bmatrix} < 0 \tag{16}$$

$$\begin{bmatrix} -M_2 & F_x \\ F_x^T & -S_4^{-1} \end{bmatrix} < 0 \tag{17}$$

where

$$\Lambda_0 = AX + XA^T - BR^{-1}B^T + \zeta D_AD_A^T + \theta D_{Ad}D_{Ad}^T,$$
$$\Lambda_1 = -S_3 + (\bar{\theta}+\bar{\lambda})E_{Ad}E_{Ad}^T,$$
$$\Lambda_2 = S_2A_{11} + A_{11}^TS_2 + YA_{21} + A_{21}^TY^T + S_4,$$
$$\Lambda_3 = -S_4,\ Y = S_2L,\ Z = [S_2\ \ Y],$$
$$G_1 = P^TS_1B, G_2 = D_A^TZ^T,\ G_3 = D_{Ad}^TZ^T,$$
$$G_4 = -ZA_d,\ G_5 = E_AX,$$
$$\Theta_0 = \frac{1}{2}(S_1(\Sigma_0 + \boldsymbol{m}_0\boldsymbol{m}_0^T) + (\Sigma_0 + \boldsymbol{m}_0\boldsymbol{m}_0^T)^TS_1),$$
$$\Theta_1 = \frac{1}{2}(S_2\Sigma_{11} + \Sigma_{11}S_2),\ \Theta_2 = \frac{1}{2}(Y\Sigma_{21} + \Sigma_{21}^TY^T),$$
$$\Theta_3 = \frac{1}{2}(Y^T\Sigma_{12} + \Sigma_{12}^TY),\ \Sigma_{22}^{1/2} = [\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_m],$$
$$\Sigma_0 = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix},\ \boldsymbol{e}_{ik} = \begin{bmatrix} 0_{k-1}^T\ 1\ 0_{i-k}^T \end{bmatrix}^T,$$
$$E_x = hI_n,\ F_x = hI_{n-m},$$

has a set of solutions $S_1 > 0$, $S_2 > 0$, $S_3 > 0$, $S_4 > 0$, $M_x > 0$, $X > 0$, Y, Z, $\zeta > 0$, $\theta > 0$, $\bar{\theta} > 0$, $\omega > 0$, $\bar{\omega} > 0$, $\bar{\lambda} > 0$, γ_0, γ_1, γ_2, γ_3, γ_4 which satisfy the relation $\bar{\theta} = \theta^{-1}$, $\bar{\omega} = \omega^{-1}$, $X = S_1^{-1}$ and $M_x = S_3^{-1}$, then the minimal order observer-based control law (8)-(10) with (12) is a guaranteed cost controller which gives the minimum expected value of the guaranteed cost

$$E[J^*] = E\left[\boldsymbol{w}^T(0)\begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}\boldsymbol{w}(0)\right] + E\left[\int_{-h}^{0}\boldsymbol{w}^T(s)\begin{bmatrix} S_3 & 0 \\ 0 & S_4 \end{bmatrix}\boldsymbol{w}(s)ds\right]$$
$$= \min\{\gamma_0+\gamma_1+\gamma_2+\gamma_3+\gamma_4+tr(M_1) + tr(M_2)\} \tag{18}$$

where $\boldsymbol{\xi}(t) = \boldsymbol{z}(t) - T\boldsymbol{x}(t)$ is the estimated error of the minimal order observer, and $\boldsymbol{w}(\cdot) = \begin{bmatrix} \boldsymbol{x}^T(\cdot) & \boldsymbol{\xi}^T(\cdot) \end{bmatrix}^T$.

Remark 1. Since (13) has a constraint of the inverse relations, an iterative algorithm via LMI approach is introduced to solve [5].

Before giving a proof of theorem 1, a key lemma is introduced.

Lemma 1. [3] Given matrices D and E of appropriate dimensions, and $F(t)$ be a matrix function satisfying $F(t)^TF(t) \le I$, then for any $\alpha > 0$, the following inequality holds

$$DF(t)E + E^TF(t)^TE^T \le \alpha DD^T + \alpha^{-1}E^TE.$$

Proof of Theorem 1.
Equations (1)-(3) and (8)-(10) yield the closed-loop system

$$\begin{bmatrix} \dot{\boldsymbol{x}}(t) \\ \dot{\boldsymbol{\xi}}(t) \end{bmatrix} = \begin{bmatrix} \Phi_1 & \Phi_2 & \Phi_3 \\ \Phi_4 & \Phi_5 & \Phi_6 \end{bmatrix}\begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{x}(t-h) \\ \boldsymbol{\xi}(t) \end{bmatrix} \tag{19}$$

where

$$\Phi_1 = A + \Delta A(t) + BK,\ \Phi_2 = A_d + \Delta A_d(t)$$
$$\Phi_3 = BKP,\ \Phi_4 = -T\Delta A(t)$$
$$\Phi_5 = -T(A_d + \Delta A_d(t)),\ \Phi_6 = D$$

Consider a candidate of Lyapunov function as

$$V(t) = V_1(t) + V_2(t) \tag{20}$$

where

$$V_1(t) = \boldsymbol{w}^T(t) \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \boldsymbol{w}(t),$$

$$V_2(t) = \int_{t-h}^{t} \boldsymbol{w}^T(t) \begin{bmatrix} S_3 & 0 \\ 0 & S_4 \end{bmatrix} \boldsymbol{w}(t)\, dt$$

then, the time derivative of (20) along to (19) is calculated in the following equation

$$\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) \tag{21}$$

where

$$\dot{V}_1(t) = \dot{\boldsymbol{w}}^T(t) \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \boldsymbol{w}(t) + \boldsymbol{w}^T(t) \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \dot{\boldsymbol{w}}(t)$$

$$\dot{V}_2(t) = \boldsymbol{w}^T(t) \begin{bmatrix} S_3 & 0 \\ 0 & S_4 \end{bmatrix} \boldsymbol{w}(t) - \boldsymbol{w}^T(t-h) \begin{bmatrix} S_3 & 0 \\ 0 & S_4 \end{bmatrix} \boldsymbol{w}(t-h)$$

By introducing $\bar{z}(t)$ and Ω, the equation (21) can be written as

$$\dot{V}(t) = \bar{\boldsymbol{z}}^T(t)\Omega\bar{\boldsymbol{z}}(t) - (\boldsymbol{x}^T(t)Q\boldsymbol{x}(t) + \boldsymbol{u}^T(t)R\boldsymbol{u}(t)) \tag{22}$$

where

$$\bar{\boldsymbol{z}}(t) = \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{x}(t-h) \\ \boldsymbol{\xi}(t) \\ \boldsymbol{\xi}(t-h) \end{bmatrix},\ \Omega = \begin{bmatrix} \Lambda_1 & \Lambda_2 & \Lambda_3 & 0 \\ \Lambda_2^T & \Lambda_4 & \Lambda_5 & 0 \\ \Lambda_3^T & \Lambda_5^T & \Lambda_6 & 0 \\ 0 & 0 & 0 & \Lambda_7 \end{bmatrix},$$

$$\begin{aligned} \Lambda_1 &= S_1(A + \Delta A(t) + BK) + (A + \Delta A(t) \\ &\quad +(BK))^T S_1 + S_3 + Q + K^T RK, \\ \Lambda_2 &= S_1(A_d + \Delta A_d(t)),\ \Lambda_3 = -\Delta A^T(t)T^T S_2 \\ &\quad +S_1 BKP,\ \Lambda_4 = -S_3, \\ \Lambda_5 &= -(A_d + \Delta A_d(t))^T T^T S_2, \\ \Lambda_6 &= S_2 D + D^T S_2,\ \Lambda_7 = -S_4. \end{aligned}$$

Under the condition

$$\Omega < 0 \tag{23}$$

the equation (22) leads to

$$\dot{V}(t) < -(\boldsymbol{x}^T(t)Q\boldsymbol{x}(t) + \boldsymbol{u}^T(t)R\boldsymbol{u}(t)) < 0 \tag{24}$$

for any $\boldsymbol{x}(t) \neq \mathbf{0}$ and the closed-loop system is asymptotically stable.

Here, the condition (23) is investigated below. By applying Lemma 1 to (23), it follows for any $\zeta > 0$, $\theta > 0$, $\omega > 0$, and $\lambda > 0$ that

$$\begin{aligned} 2\boldsymbol{x}^T(t)S_1\Delta A(t)\boldsymbol{x}(t) &= 2\boldsymbol{x}^T(t)S_1 D_A F_A(t) E_A \boldsymbol{x}(t) \\ &\leq \zeta \boldsymbol{x}^T(t) S_1 D_A D_A^T S_1 \boldsymbol{x}(t) \\ &\quad +\zeta^{-1}\boldsymbol{x}^T(t) E_A^T E_A \boldsymbol{x}(t) \end{aligned} \tag{25}$$

$$\begin{aligned} &2\,\boldsymbol{x}^T(t)S_1\Delta A_d(t)\boldsymbol{x}(t-h) \\ &= 2\boldsymbol{x}^T(t)S_1 D_{Ad} F_{Ad}(t) E_{Ad}\boldsymbol{x}(t-h) \\ &\leq \theta\boldsymbol{x}^T(t) S_1 D_{Ad} D_{Ad}^T S_1 \boldsymbol{x}(t) \\ &\quad + \theta^{-1}\boldsymbol{x}^T(t-h(t)) E_{Ad}^T E_{Ad}\boldsymbol{x}(x-h) \end{aligned} \tag{26}$$

$$\begin{aligned} &-2\,\xi^T(t)S_2 T\Delta A(t)\boldsymbol{x}(t) \\ &\quad = -2\xi^T(t)S_2 T D_A F_A(t) E_A \boldsymbol{x}(t) \\ &\quad \leq \omega\boldsymbol{\xi}^T(t) S_2 T D_A D_A^T T^T S_2 \boldsymbol{\xi}(t) \\ &\qquad + \omega^{-1}\boldsymbol{x}^T(t) E_A^T E_A \boldsymbol{x}(t) \end{aligned} \tag{27}$$

$$\begin{aligned} &-2\,\boldsymbol{x}^T(t-h)\Delta A_d^T(t) T^T S_2 \boldsymbol{\xi}(t) \\ &\quad = -2\boldsymbol{\xi}^T(t) S_2 T D_{Ad} F_{Ad}(t) E_{Ad}\boldsymbol{x}(t-h) \\ &\quad \leq \lambda\boldsymbol{\xi}^T(t) S_2 T D_{Ad} D_{Ad}^T T^T S_2 \boldsymbol{\xi}(t) \\ &\qquad + \lambda^{-1}\boldsymbol{x}^T(t-h) E_{Ad}^T E_{Ad}\boldsymbol{x}(t-h) \end{aligned} \tag{28}$$

Hence, if there exist positive scalars ζ, θ, ω, and λ, symmetric positive-definite matrices S_1, S_2, S_3 and S_4 which satisfy the following matrix inequality

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} & 0 \\ \Lambda_{12}^T & \Lambda_{14} & \Lambda_{15} & 0 \\ \Lambda_{13}^T & \Lambda_{15}^T & \Lambda_{16} & 0 \\ 0 & 0 & 0 & \Lambda_{17} \end{bmatrix} < 0 \tag{29}$$

where

$$\begin{aligned} \Lambda_{11} &= S_1 A + A^T S_1 - S_1 B R^{-1} B^T S_1 + Q \\ &\quad +\zeta S_1 D_A D_A^T S_1 + \zeta^{-1} E_A^T E_A + \theta S_1 D_{Ad} \\ &\quad D_{Ad}^T S_1 + \omega^{-1} E_A^T E_A,\ \Lambda_{12} = S_1 A_d, \\ \Lambda_{13} &= S_1 B R^{-1} B^T S_1 P,\ \Lambda_{14} = -S_3 \\ &\quad +(\theta^{-1} + \lambda^{-1}) E_{Ad}^T E_{Ad},\ \Lambda_{15} = -A_d^T T^T S_2, \\ \Lambda_{16} &= S_2 D + D^T S_2 + \omega S_2 T D_A D_A^T T^T S_2 \\ &\quad +\lambda S_2 T D_{Ad} D_{Ad}^T T^T S_2,\ \Lambda_{17} = -S_4 \end{aligned}$$

then (10) is a minimal order observer-based guaranteed cost control law and (18) is a guaranteed cost for the uncertain time-delay system (1)-(3).

Pre- and post-multiplying (29) by diag$(S_1^{-1},\ I,\ I,\ I)$ on both sides, denoting $X = S_1^{-1}$, $M_x = S_3^{-1}$, $Y = S_2 L$, $\bar{\theta} = \theta^{-1}$, $\bar{\omega} = \omega^{-1}$, substituting $D = A_{11} + LA_{21}$ into (29) and using Schur Complement lead to (13). Further, integrating (24) from 0 to T yields

$$\begin{aligned} J &= \int_0^\infty (\boldsymbol{x}^T(t)Q\boldsymbol{x}(t) + \boldsymbol{u}^T(t)R\boldsymbol{u}(t))dt \\ &< \boldsymbol{x}^T(0)S_1\boldsymbol{x}(0) + \boldsymbol{\xi}^T(0)S_2\boldsymbol{\xi}(0) \\ &\quad + \int_{-h}^0 \boldsymbol{x}^T(s)S_3\boldsymbol{x}(s)ds + \int_{-h}^0 \boldsymbol{\xi}^T(s)S_4\boldsymbol{\xi}(s)ds \\ &= J^* \end{aligned} \tag{30}$$

where J^* denotes the guaranteed cost. Here, we consider the optimal expected value of the guaranteed cost. It is calculated as

$$\begin{aligned} E\left[J^*\right] &= \mathrm{tr} S_1 E\left[\boldsymbol{x}(0)\boldsymbol{x}^T(0)\right] + \mathrm{tr} S_2 E\left[\boldsymbol{\xi}(0)\boldsymbol{\xi}^T(0)\right] \\ &\quad +\mathrm{tr} S_3 E\left[\int_{-h}^0 \boldsymbol{x}(s)\boldsymbol{x}^T(s)ds\right] \\ &\quad +\mathrm{tr} S_4 E\left[\int_{-h}^0 \boldsymbol{\xi}(s)\boldsymbol{\xi}^T(s)ds\right] \end{aligned} \tag{31}$$

A relation between mean $\boldsymbol{m}(t)$ and covariance $\Sigma(t)$ of $\boldsymbol{x}(t)$ is given by

$$\Sigma(t) = E\left[\boldsymbol{x}(t)\boldsymbol{x}^T(t)\right] - \boldsymbol{m}(t)\boldsymbol{m}^T(t) \tag{32}$$

Substituting (32) into (31) results in

$$E[J^*] = \mathrm{tr}S_1(\Sigma_0 + \boldsymbol{m}_0\boldsymbol{m}_0^T) + \mathrm{tr}S_2E\left[(\boldsymbol{z}(0) - T\boldsymbol{x}(0))(\boldsymbol{z}(0) - T\boldsymbol{x}(0))^T\right] + \mathrm{tr}S_3\int_{-h}^{0}(\Sigma(s) + \boldsymbol{m}(s)\boldsymbol{m}^T(s))\,ds + \mathrm{tr}S_4E\left[\int_{-h}^{0}\boldsymbol{\xi}(s)\boldsymbol{\xi}^T(s)ds\right] \quad (33)$$

Here, it is readily seen from (33) that

$$E\left[(\boldsymbol{z}(0) - T\boldsymbol{x}(0))(\boldsymbol{z}(0) - T\boldsymbol{x}(0))^T\right] = T\Sigma_0T^T + (\boldsymbol{z}(0) - T\boldsymbol{m}_0)(\boldsymbol{z}(0) - T\boldsymbol{m}_0)^T, \quad (34)$$

$$\mathrm{tr}S_3\int_{-h}^{0}(\Sigma(s) + \boldsymbol{m}(s)\boldsymbol{m}^T(s))ds = \mathrm{tr}S_3E_xE_x^T, \quad (35)$$

and

$$\mathrm{tr}S_4E\left[\int_{-h}^{0}\boldsymbol{\xi}^T(s)\boldsymbol{\xi}(s)ds\right] = \mathrm{tr}S_4E\left[\int_{-h}^{0}(\boldsymbol{z}(s) - T\boldsymbol{x}(s))(\boldsymbol{z}(s) - T\boldsymbol{x}(s))^Tds\right]. \quad (36)$$

It is assumed for $t \in [-h, 0]$,

$$\boldsymbol{z}(t) - T\boldsymbol{m}(t) = 0, \quad (37)$$

Therefore, (36) can be re-expressed as

$$\mathrm{tr}S_4E\left[\int_{-h}^{0}T\Sigma(s)T^Tds\right] = \mathrm{tr}S_4TE\left[\int_{-h}^{0}\Sigma(s)ds\right]T^T = \mathrm{tr}S_4TF_xF_x^TT^T \quad (38)$$

for $s = 0$, then $\Sigma(0) = \Sigma_0$ in which

$$\Sigma_0 = \begin{bmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{bmatrix}$$

Next, consider positive scalars $\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4$, $\mathrm{tr}(M_1)$, and $\mathrm{tr}(M_2)$ satisfying the following inequalities

$$\mathrm{tr}S_1(\Sigma_0 + \boldsymbol{m}_0\boldsymbol{m}_0^T) < \gamma_0 \quad (39)$$

$$\mathrm{tr}S_2\Sigma_{11} < \gamma_1 \quad (40)$$

$$\mathrm{tr}S_2L\Sigma_{21} < \gamma_2 \quad (41)$$

$$\mathrm{tr}S_2\Sigma_{12}L^T < \gamma_3 \quad (42)$$

$$\mathrm{tr}S_2L\Sigma_{22}L^T < \gamma_4 \quad (43)$$

$$\mathrm{tr}S_3E_xE_x^T < \mathrm{tr}(M_1) \quad (44)$$

$$\mathrm{tr}S_4F_xF_x^T < \mathrm{tr}(M_2) \quad (45)$$

Minimizing $\gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \mathrm{tr}(M_1) + \mathrm{tr}(M_2)$ results in giving min $E[J^*]$. By recalling $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, (39)-(42) lead to (14), and (44)-(45) lead to (16)-(17), respectively. Next, by denoting $\Sigma_{22}^{1/2} = [\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_m]$, equation (43) is calculated as

$$\mathrm{tr}S_2L\Sigma_{22}L^T = \boldsymbol{v}_1^TY^TS_2^{-1}Y\boldsymbol{v}_1 + \boldsymbol{v}_2^TY^TS_2^{-1}Y\boldsymbol{v}_2 + \cdots + \boldsymbol{v}_m^TY^TS_2^{-1}Y\boldsymbol{v}_m = \left[\boldsymbol{v}_1^TY^T\ \boldsymbol{v}_2^TY^T \cdots \boldsymbol{v}_m^TY^T\right]S_2^{-1}\begin{bmatrix}Y\boldsymbol{v}_1\\ Y\boldsymbol{v}_2\\ \vdots\\ Y\boldsymbol{v}_m\end{bmatrix} < \gamma_4 \quad (46)$$

Further, Schur complement derives (15) from (46).

□

It is noted that the inequalities (13) cannot be solved directly by LMI because they contain the scalars θ, $\bar{\theta}$, ω, $\bar{\omega}$ and matrices S_1, X, S_3, M_x which satisfy the relation $S_1^{-1} = X$, $S_3^{-1} = M_x$, $\theta^{-1} = \bar{\theta}$, $\omega^{-1} = \bar{\omega}$. There are a number of algorithms available in literature, a cone complementarity linearization approach [5], a sequential linear programming matrix method (SLPMM) [6], a Min-Max algorithm, an alternating projection method and so on to solve this kind of the problems. Here, we apply the cone complementarity linearization approach and propose the algorithm as follows.

Algorithm 1.

0: Set k_{max}, γ_{min} and κ.

1: Choose a sufficiently large initial γ such that there exists a feasible solution to LMI conditions

$$\begin{bmatrix}S_1 & I\\ I & X\end{bmatrix} > 0,\ \begin{bmatrix}S_3 & I\\ I & M_x\end{bmatrix} > 0,$$
$$\gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \mathrm{tr}(M_1) + \mathrm{tr}(M_2) < \gamma,$$
$\theta\bar{\theta} > 1$, $\omega\bar{\omega} > 1$, inequalities (13)-(17)

2.1 : Set $\bar{\gamma} = \gamma$, $k = 0$, $i = 1$, set $S_1(k) = S_1$, $X(k) = X$, $S_3(k) = S_3$, $M_x(k) = M_x$, $\theta(k) = \theta$, $\bar{\theta}(k) = \bar{\theta}$, $\omega(k) = \omega$, $\bar{\omega}(k) = \bar{\omega}$.

2.2 : Solve the following LMI problem
t_k=Minimize (trace $[S_1(k)X + X(k)S_1 + S_3(k)M_x + M_x(k)S_3) + \theta(k)\bar{\theta} + \theta\bar{\theta}(k) + \omega(k)\bar{\omega} + \omega\bar{\omega}(k)$
subject to

$$\begin{bmatrix}S_1 & I\\ I & X\end{bmatrix} > 0,\ \begin{bmatrix}S_3 & I\\ I & M_x\end{bmatrix} > 0,$$
$$\gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \mathrm{tr}(M_1) + \mathrm{tr}(M_2) < \gamma,$$
$\theta\bar{\theta} > 1$, $\omega\bar{\omega} > 1$, inequalities (13)-(17)

3.1 : If $k < k_{max}$ and $t_k > 4n+4+\kappa$ then set $k = k+1$ and go to 2.2.

3.2 : If $k \le k_{max}$, $4n + 4 \le t_k \le 4n + 4 + \kappa$, LMI conditions are satisfied, and $\gamma(0.5)^i > \gamma_{min}$ then $\bar{\gamma}$=$\bar{\gamma}$-$\gamma(0.5)^i$. Else if $\gamma(0.5)^i \le \gamma_{min}$ then exit and $\bar{\gamma}$ is an optimal value.

3.3 : If $k < k_{max}$, $4n + 4 \le t_k \le 4n + 4 + \kappa$, LMI conditions are not satisfied, $i \ne 1$ and $\gamma(0.5)^i > \gamma_{min}$ then $\bar{\gamma}$=$\bar{\gamma}$+$\gamma(0.5)^i$. Else if $\gamma(0.5)^i \le \gamma_{min}$ then exit and $\bar{\gamma}$ is an optimal value. Else if $i = 1$ then exit and no optimal solution is obtained.

3.4 : If k=k_{max}, t_k>$4n+4+\kappa$, $i \ne 1$ and $\gamma(0.5)^i > \gamma_{min}$ then $\bar{\gamma}$=$\bar{\gamma}$+$\gamma(0.5)^i$. Else if $\gamma(0.5)^i \le \gamma_{min}$ then exit and $\bar{\gamma}$ is an optimal value. Else if $i = 1$ then exit and no optimal solution is obtained.

4 : Set $i = i + 1$ and return to 3.1.

Remark 2. The algorithm used in [7] applied the decreasing of an initial value to some extent. By utilizing of an arbitrary increment $\pm\gamma(0.5)^i$, our proposed algorithm leads to the faster computation in finding an optimal value of guaranteed cost.

4. A NUMERICAL EXAMPLE

Consider a system (1) with the following parameters

$$A = \begin{bmatrix} -3 & 0 & -2 & 0 \\ 0 & -2 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 3 \\ 2 \\ -6 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} O_2 & I_2 \end{bmatrix}, \boldsymbol{m}(0) = 0_4, \; \Sigma(0) = I_4, \; R = 9,$$

$$Q = \mathrm{diag}(7, 15, 1, 3), \; D_A = \begin{bmatrix} 0.1I_2 & O_2 \\ O_2 & O_2 \end{bmatrix},$$

$$E_A = \begin{bmatrix} 0.3I_2 & 0.3I_2 \\ O_2 & O_2 \end{bmatrix}, A_d = 0.05A, \; D_{Ad} = 0.01D_A,$$

$$E_{Ad} = 0.01E_A, \; h = 0.5.$$

Applying Theorem 1 with k_{max}=1000, γ_{min}=0.0001, κ=0.0001 and initial γ=100, there exist symmetric positive definite matrices

$$S_1 = \begin{bmatrix} 1.0918 & 0.0317 & 0.0391 & -0.0053 \\ 0.0317 & 4.4202 & 1.1652 & 1.1568 \\ 0.0391 & 1.1652 & 1.1077 & 1.1052 \\ -0.0053 & 1.1568 & 1.1052 & 7.5582 \end{bmatrix},$$

$$X = \begin{bmatrix} 0.9174 & 0.0027 & -0.0416 & 0.0063 \\ 0.0027 & 0.3130 & -0.3297 & 0.0003 \\ -0.0416 & -0.3297 & 1.4058 & -0.1551 \\ 0.0063 & 0.0003 & -0.1551 & 0.1549 \end{bmatrix},$$

$$S_2 = \begin{bmatrix} 0.1753 & 0.2057 \\ 0.2057 & 0.2413 \end{bmatrix},$$

$$S_3 = \begin{bmatrix} 0.1585 & 0.0745 & 0.0651 & -0.0241 \\ 0.0745 & 0.6296 & 0.0342 & 0.1254 \\ 0.0651 & 0.0342 & 0.0671 & -0.0014 \\ -0.0241 & 0.1254 & -0.0014 & 0.1162 \end{bmatrix},$$

$$M_x = \begin{bmatrix} 12.2357 & -1.7002 & -10.9128 & 4.2397 \\ -1.7002 & 2.3406 & 0.3961 & -2.8752 \\ -10.9128 & 0.3961 & 25.2327 & -2.3782 \\ 4.2397 & -2.8752 & -2.3782 & 12.5643 \end{bmatrix},$$

$$S_4 = \begin{bmatrix} 0.0014 & 0.0081 \\ 0.0081 & 0.0490 \end{bmatrix},$$

and scalars

$$\theta = 0.9912, \bar{\theta} = 1.0089, \; \omega = 8.3945, \bar{\omega} = 0.1191$$

which satisfy LMIs in (13)-(17) and inverse relations.

Further, we obtain a solution

$$L = \begin{bmatrix} -0.1591 & -0.0133 \\ -0.0097 & -0.2325 \end{bmatrix}, \; E = \begin{bmatrix} -2.5026 & -0.0451 \\ -0.0233 & -1.5191 \end{bmatrix},$$

$$W = \begin{bmatrix} 0.1591 & 0.0133 \\ 0.0097 & 0.2325 \\ 1.0000 & 0 \\ 0 & 1.0000 \end{bmatrix},$$

$$K = \begin{bmatrix} -0.3443 & -0.3445 & 0.3437 & -0.3583 \end{bmatrix},$$

$$\bar{\gamma} = E[J^*] = 14.8699.$$

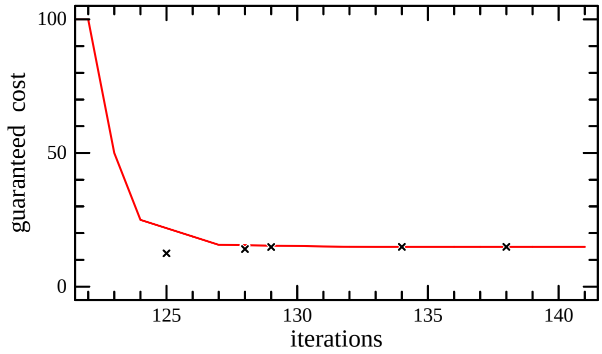

Fig. 1. Trajectories of the guaranteed cost. The marks **x** show that a feasible solution cannot be obtained for the guaranteed cost candidate $\bar{\gamma}$ and the optimal performance index is greater than **x**.

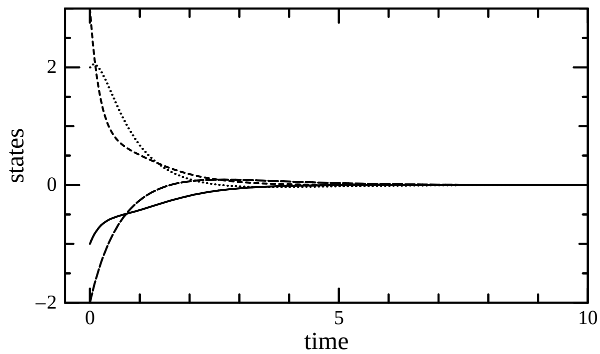

Fig. 2. Trajectories of states x_1(—), x_2(- -), x_3($\cdots$) and x_4(-·-).

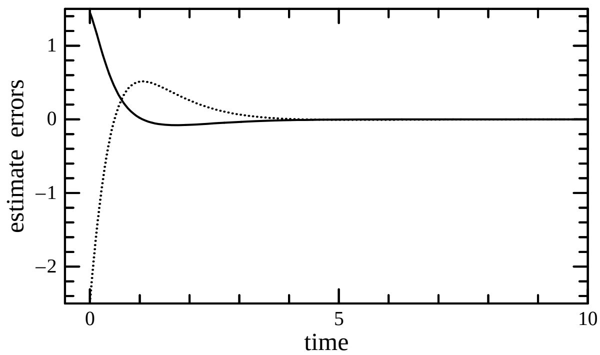

Fig. 3. Trajectories of estimate errors ξ_1(—) and ξ_2(- -).

For $F_A(t) = F_{Ad}(t) = I$, the closed loop system poles are -4.7042, $-2.8800 \pm 1.7538i$, and -2.1626. Trajectories of states and estimate errors are depicted in Figs. 2-3 with $x(0) = [-1, 2, 3, -2]^T$.

5. CONCLUSION

This paper discusses a minimal order observer-based guaranteed cost control problem for time-delay systems.

A sufficient condition for the existence of state feedback guaranteed cost controllers is derived on the basis of the LMI feasible solutions. The optimal cost control is provided by means of the minimizing the upper bound of guaranteed cost control. A numerical example is given to illustrate the proposed method.

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