A full order observer-based guaranteed cost controller for uncertain linear systems

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Abstract: In this paper, a design scheme of a full order observer-based guaranteed cost controller for uncertain linear systems, in which all state variables cannot be known, is presented. The perturbations in system states and input are assumed to be described by structural uncertainties. An iterative linear matrix inequality (ILMI) approach is used to design the observer-based controller since the problems contain inverse relations. A numerical example is given to illustrate the proposed method.

Keywords: a full order observer-based control; guaranteed cost; iterative linear matrix inequality; ILMI.

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1 Introduction

Considerable attention to the problem in robust stability analysis and robust stabilisation of uncertain systems has been attracting many authors for several last decades, e.g., (Chang and Peng, 1972; Petersen and McFarlane, 1994; Sato et al., 2008) and references therein. These uncertainties occur and may be caused by unknown noises, environmental effects and change of parameters. There has been much effort to design a controller which not only achieves the stability of the uncertain system but also guarantees an adequate level of performance. One approach to this problem is the guaranteed cost control method via linear matrix inequality (LMI) technique (Lien, 2004; Won and Park, 1999), which is a powerful tool in the control theory and applications in recent years (Matsuo et al., 2008). Moreover, the guaranteed cost control approach has been recently extended to the time-delay systems (Chen et al., 2004; Esfahani and Petersen, 1998).

Though the controller is usually constructed by using state variables, it may not be possible to measure all the states of the system in many cases due to cost problems, difficulties in measurement and uncertainty perturbations. Thus, the observer-based control is probably well suited and better than the state control feedback in such situations. In addition, the problem of designing an observer-based guaranteed cost controller has received some attention in recent years (Lien, 2005). Therefore, we consider a state observer because of its ability to reconstruct the state of a dynamic system. In the observer-based guaranteed cost control problem, finite solutions cannot be usually obtained because the loss of the cost depends on the unknown initial state variables as well as the case without uncertainties (Miller, 1973). The work of Mahmoud and Zribi (2003) implicitly needs an initial value of the states and error states to obtain the guaranteed cost value. Instead of this, we assume that their mean and covariance are known. This approach is theoretically more acceptable for systems with an observer and has more practical sense.

Further, the restriction on the form of the observer gain matrix should be used in the guaranteed cost controllers with a full order observer. We can see the restriction from those of Mahmoud and Zribi (2003), Won and Park (1999), and Lien (2005). Otherwise, they may have some poles with an infinitely small negative real parts when the observer gain is left free (Ishitobi and Miyachi, 2008). Our study does not put the restriction on the form of the observer gain. Firstly, we design a minimal order observer-based guaranteed cost controller and extend it to a full order observer-based without any restriction to the observer gain matrix of the minimal order observer-based, but it is left free.

Since inverse relations of variables appear, this paper concerns a design method of a full order observer-based guaranteed cost controller via an iterative linear matrix inequality (ILMI) technique such that a feasible solution to the convex LMI problems will be achieved iteratively. An assumption on the statistical properties of the unknown initial state variables for a full order observer-based guaranteed cost problem is used.

Outline of this paper is as follows. Section 2 states the problem of a continuous time uncertain system with statistical properties. Section 3 provides the main results in LMI formulations, involving an observer-based controller design algorithm. Section 4 gives a numerical example to illustrate the proposed method and finally, Section 5 presents the conclusion of this work.

2 Problem statement

Consider the following continuous-time uncertain system of the form

$$\dot{\boldsymbol{x}}(t) = \left(A + \Delta A(t)\right)\boldsymbol{x}(t) + \left(B + \Delta B(t)\right)\boldsymbol{u}(t) \tag{1}$$

$$\boldsymbol{y}(t) = C\boldsymbol{x}(t) \tag{2}$$

where $\boldsymbol{x}(t) \in \Re^n$ is the state vector, $\boldsymbol{u}(t) \in \Re^r$ is the control input vector, $y(t) \in \Re^m$ is the measured output vector, A, B, C are known constant real-valued matrices with appropriate dimensions, and C is restricted to the form of $C = [O\ I_m]$. Matrices $\Delta A(t)$ and $\Delta B(t)$ denote real-valued matrix functions representing parameter uncertainties in the system state and input. It is assumed that the uncertain matrices are represented by

$$\Delta A(t) = D_A F_A(t) E_A,\ \Delta B(t) = D_B F_B(t) E_B \tag{3}$$

with relations satisfying

$$F_A^T(t) F_A(t) \le I,\ F_B^T(t) F_B(t) \le I$$

where D_A, D_B, E_A, E_B are constant real-valued known matrices with appropriate dimensions, and $F_A(t)$ and $F_B(t)$ are real-time-varying unknown continuous and deterministic matrices. The assumption above on F_A and F_B leads to the adoption of a constant Lyapunov function below though the result may be conservative.

We further assume that the initial state variable $\boldsymbol{x}(0)$ is unknown, but their mean and covariance are known, equivalently (Miller, 1973)

$$E\left[\boldsymbol{x}(0)\right] = \boldsymbol{m}_0 \tag{4}$$

$$E\left[\left(\boldsymbol{x}(0) - \boldsymbol{m}_0\right)\left(\boldsymbol{x}(0) - \boldsymbol{m}_0\right)^T\right] = \Sigma_0 > O \tag{5}$$

where $E[\cdot]$ denotes the expected value operator.

The problem considered here is to design a full order observer

$$\dot{\hat{\boldsymbol{x}}}(t) = \left(A - K_o C\right)\hat{\boldsymbol{x}}(t) + B\boldsymbol{u}(t) + K_o \boldsymbol{y}(t) \tag{6}$$

and a controller

$$\boldsymbol{u}(t) = -K\hat{\boldsymbol{x}}(t) \tag{7}$$

so that achieving an upper bound on the following quadratic performance index

$$E[J] = E\left[\int_0^\infty \left(\boldsymbol{x}^T(t) Q \boldsymbol{x}(t) + \boldsymbol{u}^T(t) R \boldsymbol{u}(t)\right) dt\right] \tag{8}$$

associated with the uncertain system (1) and (2) where Q and R are given symmetric positive-definite matrices.

3 Main results

In this section, a sufficient condition is established for the existence of a full order observer-based guaranteed cost controller for the uncertain systems (1) and (2). The introduction of the restriction on the feedback controller gain matrix in the form below results in the formulations of the algorithm with LMIs.

$$K = R^{-1} B^T S_1 \tag{9}$$

where S_1 is a symmetric positive-definite matrix. Further, we assume that

$$\hat{\boldsymbol{x}}(0) - \boldsymbol{m}_0 = 0 \tag{10}$$

The main result of this study is given by Theorem 1.

Theorem 1: If the following matrix inequalities optimisation problem; min $\{\gamma_1 + \gamma_2\}$ subject to

$$\begin{bmatrix} \Lambda_1 & X^T & XE_A^T & XE_A^T \\ X & -Q^{-1} & 0 & 0 \\ E_A X & 0 & -\delta I & 0 \\ E_A X & 0 & 0 & -\epsilon I \end{bmatrix} < 0 \tag{11}$$

$$\begin{bmatrix} \Lambda_2 & \Lambda_3 & \Lambda_4 & \Lambda_5 & \Lambda_6 & \Lambda_7 & \Lambda_8 \\ \Lambda_3^T & -R & 0 & 0 & 0 & 0 & 0 \\ \Lambda_4^T & 0 & -\epsilon_{inv} I & 0 & 0 & 0 & 0 \\ \Lambda_5^T & 0 & 0 & -\mu I & 0 & 0 & 0 \\ \Lambda_6^T & 0 & 0 & 0 & -\omega_{inv} I & 0 & 0 \\ \Lambda_7^T & 0 & 0 & 0 & 0 & -\tau_{inv} I & 0 \\ \Lambda_8^T & 0 & 0 & 0 & 0 & 0 & -\tau I \end{bmatrix} < 0 \tag{12}$$

$$\sum_{k=1}^{n} e_{nk}^T \Theta_1 e_{nk} < \gamma_1 \tag{13}$$

$$\sum_{k=1}^{n} e_{nk}^T \Theta_2 e_{nk} < \gamma_2 \tag{14}$$

where

$$\begin{aligned} \Lambda_1 &= AX + XA^T - BR^{-1}B^T + \delta D_A D_A^T + \rho D_B D_B^T \\ &\quad + \rho^{-1} BR^{-1} E_B^T E_B R^{-1} B^T + \mu D_B D_B^T \\ &\quad + \omega^{-1} BR^{-1} E_B^T E_B R^{-1} B^T \\ \Lambda_2 &= S_2 A + (S_2 A)^T + S_2 \bar{I} \Lambda C + (S_2 \bar{I} \Lambda C)^T \\ &\quad + S_2 \bar{L} \Lambda C + (S_2 \bar{L} \Lambda C)^T - S_2 A \bar{I} C \\ &\quad - (S_2 A \bar{I} C)^T - S_2 A \bar{L} C - (S_2 A \bar{L} C)^T \\ \Lambda_3 &= S_1 B,\ \Lambda_4 = S_2 D_A \\ \Lambda_5 &= \Lambda_8 = S_1 B R^{-1} E_B^T \\ \Lambda_6 &= \Lambda_7 = S_2 D_B \\ \bar{I} &= \begin{bmatrix} O \\ I \end{bmatrix}, \bar{L} = \begin{bmatrix} -L \\ O \end{bmatrix}, e_{nk} = \begin{bmatrix} \mathbf{0}_{k-1}^T & 1 & \mathbf{0}_{n-k}^T \end{bmatrix}^T \\ \Theta_1 &= \frac{1}{2}\left(\left(S_1\left(\Sigma_0 + \boldsymbol{m}_0 \boldsymbol{m}_0^T\right) + \left(\Sigma_0 + \boldsymbol{m}_0 \boldsymbol{m}_0^T\right)^T S_1\right)\right) \\ \Theta_2 &= \frac{1}{2}\left(S_2 \Sigma_0 + \Sigma_0 S_2\right) \end{aligned}$$

has a solution $S_1 > 0$, $S_2 > 0$, $X > 0$, L, $\delta > 0$, $\epsilon > 0$, $\epsilon_{inv} > 0$, $\mu > 0$, $\omega_{inv} > 0$, $\tau > 0$, $\tau_{inv} > 0$, $\rho > 0$, $\rho_{inv} > 0$, γ_1, γ_2 which satisfy the relation $\epsilon^{-1} = \epsilon_{inv}$, $\tau^{-1} = \tau_{inv}$, $\rho^{-1} = \rho_{inv}$ and $S_1^{-1} = X$, then the full order observer-based control law (7) with (9) is a guaranteed cost controller which gives the minimum expected value of the guaranteed cost

$$E\left[J^*\right] = E\left[\boldsymbol{x}^T(0) S_1 \boldsymbol{x}(0) + \boldsymbol{e}^T(0) S_2 \boldsymbol{e}(0)\right] \tag{15}$$

where $\boldsymbol{e}(t) = \hat{\boldsymbol{x}}(t) - \boldsymbol{x}(t)$ is the estimated error of the full order observer.

Remark 1: Since (11) and (12) have a constraint of the relationship of the inverse, ILMI approach is introduced to solve the problem (Ghaoui et al., 1997; Cao et al., 1998).

Before giving a proof of Theorem 1, a key lemma is introduced.

Lemma 1 (Mahmoud and Zribi, 2003): Let D and E be matrices of appropriate dimensions, and F be a matrix function satisfying $F^T F \le$ I. Then for any positive scalar α, the following inequality holds

$$DFE + E^T F^T D^T \le \alpha DD^T + \alpha^{-1} E^T E. \tag{16}$$

Proof of Theorem 1:

Equations (1), (6) and (7) yield the closed-loop system

$$\begin{bmatrix} \dot{\boldsymbol{x}}(t) \\ \dot{\boldsymbol{e}}(t) \end{bmatrix} = \begin{bmatrix} \Phi_1 & \Phi_2 \\ \Phi_3 & \Phi_4 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{e}(t) \end{bmatrix} \tag{17}$$

where

$$\begin{aligned} \Phi_1 &= A + \Delta A(t) + \left(B + \Delta B(t)\right) K \\ \Phi_2 &= \left(B + \Delta B(t)\right) K \\ \Phi_3 &= -\Delta A(t) - \Delta B(t) K \\ \Phi_4 &= A - K_o C - \Delta B(t) K \end{aligned}$$

Although we can see from (1) and (2) that the system is time varying, according to (3), it is sufficient to define a candidate of Lyapunov function as

$$V(t) = \boldsymbol{x}^T(t) S_1 \boldsymbol{x}(t) + \boldsymbol{e}^T(t) S_2 \boldsymbol{e}(t) \tag{18}$$

Then, the time derivative of (18) along to (17) is calculated as

$$\begin{aligned} \dot{V}(t) &= 2\boldsymbol{x}^T(t) S_1 \dot{\boldsymbol{x}}(t) + 2\boldsymbol{e}^T(t) S_2 \dot{\boldsymbol{e}}(t) \\ &= 2\boldsymbol{x}^T(t) S_1 \Big\{\Big(A + \Delta A(t) + \left(B + \Delta B(t)\right) K\Big) \boldsymbol{x}(t) \\ &\qquad + \left(-\Delta A(t) - \Delta B(t) K\right) \boldsymbol{e}(t)\ \Big\} \\ &\qquad + 2\boldsymbol{e}(t) S_2 \left\{\left(-\Delta A(t) - \Delta B(t) K\right) \boldsymbol{x}(t)\right. \\ &\qquad \left. + \left(A - K_o C - \Delta B(t) K\right) \boldsymbol{e}(t)\right\} \\ &= z^T(t) \Omega z(t) - \left(\boldsymbol{x}^T(t) Q \boldsymbol{x}(t) + \boldsymbol{u}^T(t) R \boldsymbol{u}(t)\right) \end{aligned} \tag{19}$$

where

$$z(t) = \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{e}(t) \end{bmatrix}, \Omega = \begin{bmatrix} \bar{\Lambda}_1 & \bar{\Lambda}_2 \\ \bar{\Lambda}_2^T & \bar{\Lambda}_3 \end{bmatrix}$$

$$\begin{aligned}\bar{\Lambda}_1 &= S_1\left(A+\Delta A(t)\right)+\left(A+\Delta A(t)\right)^T S_1+Q\\ &\quad -S_1BR^{-1}B^TS_1-S_1\Delta BR^{-1}B^TS_1\\ &\quad -S_1BR^{-1}\Delta B^TS_1\\ \bar{\Lambda}_2 &= -\Delta A^T(t)S_2-S_1\Delta BR^{-1}B^TS_1\\ &\quad +S_1BR^{-1}\Delta B^TS_2\\ \bar{\Lambda}_3 &= S_2\left(A-K_oC\right)+\left(A-K_oC\right)^T S_2+S_1BR^{-1}B^TS_1\\ &\quad +S_2\Delta BR^{-1}B^TS_1+S_1BR^{-1}\Delta B^TS_2\end{aligned}$$

Under the condition

$$\Omega<0 \tag{20}$$

equation (19) leads to

$$\dot{V}(t)<-\left(\boldsymbol{x}^T(t)Q\boldsymbol{x}(t)+\boldsymbol{u}^T(t)R\boldsymbol{u}(t)\right)<0 \tag{21}$$

for any $x(t)\neq 0$ and the closed-loop system is asymptotically stable.

Next, the condition (20) is investigated below. By applying Lemma 1 to (20), it follows for any $\delta>0$, $\epsilon>0$, $\rho>0$, $\mu>0$, $\omega>0$ and $\tau>0$ that

$$\begin{aligned}&2\boldsymbol{x}^T(t)S_1\Delta A(t)\boldsymbol{x}(t)=2\boldsymbol{x}^T(t)S_1D_AF_A(t)E_A\boldsymbol{x}(t)\\ &\le\delta\boldsymbol{x}(t)S_1D_AD_A^TS_1\boldsymbol{x}(t)+\delta^{-1}\boldsymbol{x}^T(t)E_A^TE_A\boldsymbol{x}(t)\end{aligned} \tag{22}$$

$$\begin{aligned}&-2\boldsymbol{x}^T(t)S_2\Delta A(t)\boldsymbol{x}(t)=-2\boldsymbol{x}^T(t)S_2D_AF_A(t)E_A\boldsymbol{x}(t)\\ &\le\epsilon\boldsymbol{e}^T(t)S_2D_AD_A^TS_2\boldsymbol{e}(t)+\epsilon^{-1}\boldsymbol{x}^T(t)E_A^TE_A\boldsymbol{x}(t)\end{aligned} \tag{23}$$

$$\begin{aligned}&-2\boldsymbol{x}^T(t)S_1\Delta B(t)R^{-1}B^TS_1\boldsymbol{x}(t)\\ &=-2\boldsymbol{x}^T(t)S_1D_BF_B(t)E_BR^{-1}B^TS_1\boldsymbol{x}(t)\\ &\le\rho\boldsymbol{x}^T(t)S_1D_BD_B^TS_1\boldsymbol{x}(t)\\ &+\rho^{-1}\boldsymbol{x}^T(t)S_1BR^{-1}E_B^TE_BR^{-1}B^TS_1\boldsymbol{x}(t)\end{aligned} \tag{24}$$

$$\begin{aligned}&-2\boldsymbol{x}^T(t)S_1\Delta B(t)R^{-1}B^TS_1\boldsymbol{e}(t)\\ &=-2\boldsymbol{x}^T(t)S_1D_BF_B(t)E_BR^{-1}B^TS_1\boldsymbol{e}(t)\\ &\le\mu\boldsymbol{x}^T(t)S_1D_BD_B^TS_1\boldsymbol{x}(t)\\ &+\mu^{-1}\boldsymbol{e}^T(t)S_1BR^{-1}E_B^TE_BR^{-1}B^TS_1\boldsymbol{e}(t)\end{aligned} \tag{25}$$

$$\begin{aligned}&2\boldsymbol{e}^T(t)S_2\Delta B(t)R^{-1}B^TS_1\boldsymbol{x}(t)\\ &=2\boldsymbol{e}^T(t)S_2D_BF_B(t)E_BR^{-1}B^TS_1\boldsymbol{x}(t)\\ &\le\omega\boldsymbol{e}^T(t)S_2D_BD_B^TS_2\boldsymbol{e}(t)\\ &+\omega^{-1}\boldsymbol{x}^T(t)S_1BR^{-1}E_B^TE_BR^{-1}B^TS_1\boldsymbol{x}(t)\end{aligned} \tag{26}$$

$$\begin{aligned}&2\boldsymbol{e}^T(t)S_2\Delta B(t)R^{-1}B^TS_1\boldsymbol{e}(t)\\ &=2\boldsymbol{e}^T(t)S_2D_BF_B(t)E_BR^{-1}B^TS_1\boldsymbol{e}(t)\\ &\le\tau\boldsymbol{e}^T(t)S_2D_BD_B^TS_2\boldsymbol{e}(t)\\ &+\tau^{-1}\boldsymbol{e}^T(t)S_1BR^{-1}E_B^TE_BR^{-1}B^TS_1\boldsymbol{e}(t)\end{aligned} \tag{27}$$

Hence, if there exist scalars $\delta>0$, $\epsilon>0$, $\rho>0$, $\mu>0$, $\tau>0$, a matrix K_o, symmetric positive-definite matrices S_1 and S_2 which satisfy the following matrix inequality

$$\begin{bmatrix}\Lambda_4 & \mathbf{0}\\ \mathbf{0} & \Lambda_5\end{bmatrix}<0 \tag{28}$$

where

$$\begin{aligned}\Lambda_4 &= S_1A+A^TS_1-S_1BR^{-1}B^TS_1+Q+\delta S_1D_AD_A^TS_1\\ &\quad +\delta^{-1}E_A^TE_A+\epsilon^{-1}E_A^TE_A+\rho S_1D_BD_B^TS_1\\ &\quad +\mu S_1D_BD_B^TS_1+\rho^{-1}S_1BR^{-1}E_B^TE_BR^{-1}B^TS_1\\ &\quad +\omega^{-1}S_1BR^{-1}E_B^TE_BR^{-1}B^TS_1\\ \Lambda_5 &= S_2\left(A-K_oC\right)+\left(A-K_oC\right)^T S_2+\epsilon S_2D_AD_A^TS_2\\ &\quad +S_1BR^{-1}B^TS_1+\mu^{-1}S_1BR^{-1}E_B^TE_BR^{-1}B^TS_1\\ &\quad +\tau^{-1}S_1BR^{-1}E_B^TE_BR^{-1}B^TS_1+\omega S_2D_BD_B^TS_2\\ &\quad +\tau S_2D_BD_B^TS_2\end{aligned}$$

then (7) is a full order observer-based guaranteed cost control law and (15) is a guaranteed cost for the uncertain system (1) and (2).

In this paper, a full order observer is constructed by an output estimator and a minimal order observer that estimates the state variables except for the outputs (Telford and Moore, 1989).

An output estimator is given by

$$\dot{\hat{\boldsymbol{y}}}(t)=C\left(A\hat{\boldsymbol{x}}_m(t)+B\boldsymbol{u}(t)\right)-\Lambda\left(\boldsymbol{y}(t)-\hat{\boldsymbol{y}}(t)\right) \tag{29}$$

where Λ is a stable matrix with specified eigenvalues, and a minimal order observer is designed by

$$\dot{\boldsymbol{w}}(t)=\hat{A}\boldsymbol{w}(t)+\hat{B}\boldsymbol{u}(t)+\hat{L}\boldsymbol{y}(t) \tag{30}$$

$$\hat{\boldsymbol{x}}_m(t)=\hat{C}\boldsymbol{w}(t)+\hat{D}\boldsymbol{y}(t) \tag{31}$$

where

$$\hat{A}=F_oA\hat{C} \tag{32}$$

$$\hat{B}=F_0B \tag{33}$$

$$\hat{C}=\begin{bmatrix}I\\0\end{bmatrix} \tag{34}$$

$$\hat{D}=\begin{bmatrix}-L\\I\end{bmatrix} \tag{35}$$

$$F_0=\begin{bmatrix}I & L\end{bmatrix} \tag{36}$$

$$\hat{L}=F_oA\hat{D} \tag{37}$$

and $\boldsymbol{w}(t)$ is an estimate of $F_o\boldsymbol{x}(t)$.

Here, the following relations have to be satisfied

$$F_oA=\hat{A}F_o+\hat{L}C \tag{38}$$

$$\hat{C}F_o+\hat{D}C=I_n \tag{39}$$

$$K_0=A\hat{D}-\hat{D}\Lambda \tag{40}$$

$$\Lambda=C\left(A-K_oC\right)\hat{D} \tag{41}$$

Then, it is obtained that

$$C\hat{D} = I_m \tag{42}$$

$$F_o\hat{C} = I_{n-m} \tag{43}$$

$$C\hat{C} = O_{m\times(n-m)} \tag{44}$$

$$F_o\hat{D} = O_{(n-m)\times m} \tag{45}$$

$$\begin{aligned}\hat{\boldsymbol{x}}(t) &= \hat{C}\boldsymbol{w}(t) + \hat{D}\hat{\boldsymbol{y}}(t)\\ &= \hat{\boldsymbol{x}}_m(t) + \hat{D}\left(\hat{\boldsymbol{y}}(t) + \boldsymbol{y}(t)\right)\end{aligned} \tag{46}$$

From (32) to (45), the matrix $A - K_oC$ is re-expressed as

$$\begin{aligned}&A - K_oC\\ &= \begin{bmatrix}\hat{D} & \hat{C}\end{bmatrix}\begin{bmatrix}C\\ F_o\end{bmatrix}(A-K_oC)\begin{bmatrix}\hat{D} & \hat{C}\end{bmatrix}\begin{bmatrix}C\\ F_o\end{bmatrix}\\ &= \begin{bmatrix}\hat{D} & \hat{C}\end{bmatrix}\begin{bmatrix}C(A-K_oC)\hat{D} & C(A-K_oC)\hat{C}\\ F_o(A-K_oC)\hat{D} & F_o(A-K_oC)\hat{C}\end{bmatrix}\\ &\times\begin{bmatrix}C\\ F_o\end{bmatrix}\\ &= \begin{bmatrix}\hat{D} & \hat{C}\end{bmatrix}\begin{bmatrix}\Lambda & CA\hat{C}\\ O & \hat{A}\end{bmatrix}\begin{bmatrix}C\\ F_o\end{bmatrix}\\ &= \hat{D}\Lambda C + \hat{D}CA\hat{C}F_o + \hat{C}\hat{A}F_o\\ &= \hat{D}\Lambda C + \left(I_n - \hat{C}F_o\right)A\hat{C}F_o + \hat{C}F_oA\hat{C}F_o\\ &= \hat{D}\Lambda C + A\left(I_n - \hat{D}C\right)\\ &= \hat{D}\Lambda C + A - A\hat{D}C\end{aligned} \tag{47}$$

Pre- and post-multiplying (28) by $diag(S_1^{-1}, I)$ on both sides, denoting $X = S_1^{-1}$, $\epsilon_{inv} = \epsilon^{-1}$, $\tau_{inv} = \tau^{-1}$, $\rho_{inv} = \rho^{-1}$, substituting (47) into (28) and using Schur complement (Boyd et al., 1994), lead to (11) and (12).

Further, integrating (21) from 0 to T yields

$$\begin{aligned}&\boldsymbol{x}^T(T)S_1\boldsymbol{x}(T) + \boldsymbol{e}^T(T)S_2\boldsymbol{e}(T)\\ &-\left(\boldsymbol{x}^T(0)S_1\boldsymbol{x}(0) + \boldsymbol{e}^T(0)S_2\boldsymbol{e}(0)\right)\\ &< -\int_0^T\left(\boldsymbol{x}^T(t)Q\boldsymbol{x}(t) + \boldsymbol{u}^T(t)R\boldsymbol{u}(t)\right)dt < 0\end{aligned} \tag{48}$$

Here, the asymptotic stability of the closed-loop system implies that

$$\boldsymbol{x}^T(T)S_1\boldsymbol{x}(T) \leftarrow 0, \boldsymbol{e}^T(T)S_1\boldsymbol{e}(T) \rightarrow 0 \tag{49}$$

as T tends to the infinity. Hence, it is obtained that

$$\begin{aligned}J &= \int_0^\infty\left(\boldsymbol{x}^T(t)Q\boldsymbol{x}(t) + \boldsymbol{u}^T(t)R\boldsymbol{u}(t)\right)dt\\ &< \boldsymbol{x}^T(0)S_1\boldsymbol{x}(0) + \boldsymbol{e}^T(0)S_2\boldsymbol{e}(0)\\ &= J^*\end{aligned} \tag{50}$$

where J^* denotes the guaranteed cost. Here, we consider the optimal expected value of the guaranteed cost. It is calculated as

$$\begin{aligned}E\left[J^*\right] &= E\left[\boldsymbol{x}^T(0)S_1\boldsymbol{x}(0) + \boldsymbol{e}^T(0)S_2\boldsymbol{e}(0)\right]\\ &= E\left[\mathrm{tr}S_1\boldsymbol{x}(0)\boldsymbol{x}^T(0) + \mathrm{tr}S_2\boldsymbol{e}(0)\boldsymbol{e}^T(0)\right]\\ &= \mathrm{tr}S_1E\left[\boldsymbol{x}(0)\boldsymbol{x}^T(0)\right] + \mathrm{tr}S_2E\left[\boldsymbol{e}(0)\boldsymbol{e}^T(0)\right]\end{aligned} \tag{51}$$

A relation between mean and covarience of $\boldsymbol{x}(0)$ is given by

$$\Sigma_0 = E\left(\boldsymbol{x}(0)\boldsymbol{x}^T(0)\right) - \boldsymbol{m}_0\boldsymbol{m}_0^T \tag{52}$$

Substituting (52) into (51) yields

$$\begin{aligned}E\left[J^*\right] &= \mathrm{tr}S_1\left(\Sigma_0 + \boldsymbol{m}_0\boldsymbol{m}_0^T\right)\\ &+\mathrm{tr}S_1\left\{\Sigma_0 + \left(\hat{\boldsymbol{x}}(0) - \boldsymbol{m}_0\right)\left(\hat{\boldsymbol{x}}(0) - \boldsymbol{m}_0\right)^T\right\}\end{aligned} \tag{53}$$

Then substituting (10) into (53) yields

$$E\left[J^*\right] = \mathrm{tr}S_1\left(\Sigma_0 + \boldsymbol{m}_0\boldsymbol{m}_0^T\right) + \mathrm{tr}S_2\Sigma_0 \tag{54}$$

Here, we consider positive scalars γ_1 and γ_2 satisfying the following inequalities

$$\mathrm{tr}S_1\left(\Sigma_0 + \boldsymbol{m}_0\boldsymbol{m}_0^T\right) < \gamma_1 \tag{55}$$

$$\mathrm{tr}S_2\Sigma_0 < \gamma_2 \tag{56}$$

Minimising $\gamma_1 + \gamma_2$ results in giving min $E[J^*]$. By recalling tr(AB) = tr(BA), (55) and (56) lead to (13), (14). Q.E.D.

It is noted that the inequalities (11) and (12) cannot be solved directly by LMI because they contain the scalars ϵ, ϵ_{inv}, τ, τ_{inv}, ρ, ρ_{inv} and two matrices S_1, X which have to satisfy the relation $S_1^{-1} = X$, $\epsilon^{-1} = \epsilon_{inv}$, $\tau^{-1} = \tau_{inv}$, $\rho^{-1} = \rho_{inv}$ and the two matrices S_2 and $\bar{L}$ that have to be kept constant one by one. Using an ILMI approach means that every computation which has to solve a minimisation problem keeps all obtained matrices and scalar variables as initial values for the next iterations. For the first restriction, there are a number of algorithms available in literature, a cone complementarity linearisation approach (Ghaoui et al., 1997), a sequential linear programming matrix method (SLPMM) (Leibfritz, 2001), a Min-Max algorithm, an alternating projection method and so on to solve this kind of the problems. Here, we apply the cone complementarity linearisation approach. For the second restriction, the index $\gamma_1 + \gamma_2$ is minimised by iteration with keeping one of S_2 and $\bar{L}$ constant alternately.

An algorithm is shown in the following steps.

1. Design a minimal order observer-based guaranteed cost controllers, get an optimal guaranteed cost γ_{minobs} and set the obtained gain as L_{fix} (Matsunaga et al., 2009).
2. Design a guaranteed cost controller with a full order observer under a fixed $\bar{L}$ for $\gamma(k)$ where $\gamma(1)$ is appropriately given, when k = 1, 2, ···.

   2.1 Solve a convex problem

$\begin{bmatrix} S_1 & I \\ I & X \end{bmatrix} > 0, \gamma_1 + \gamma_2 < \gamma(k), \epsilon\epsilon_{inv} > 0,$

$\tau\tau_{inv} > 0, \rho\rho_{inv} > 0$, inequalities (11) to (14),

with

$\overline{L} = \begin{bmatrix} -L_{fix} \\ O \end{bmatrix}$

If infeasible, then go to Step 4 with $\gamma = \overline{\gamma}(k-1)(k \geq 2)$

Else set

$S_1(0) = S_1, X(0) = X, \epsilon(0) = \epsilon, \epsilon_{inv}(0) = \epsilon_{inv},$
$\tau(0) = \tau, \tau_{inv}(0) = \tau_{inv}, \rho(0) = \rho, \rho_{inv}(0) = \rho_{inv}.$

2.2 When j = 1, 2, ···, solve a minimisation problem of α by LMI under

$\begin{bmatrix} S_1 & I \\ I & X \end{bmatrix} > 0, \gamma_1 + \gamma_2 < \gamma(k), \epsilon\epsilon_{inv} > 0,$

$\tau\tau_{inv} > 0, \rho\rho_{inv} > 0,$

$\mathrm{tr}[S_1(j)X + X(j)S_1] + \epsilon(j)\epsilon_{inv} + \epsilon\epsilon_{inv}(j) + \ldots$
$\tau(j)\tau_{inv} + \tau\tau_{inv}(j) + \rho(j)\rho_{inv} + \rho\rho_{inv}(j) < \alpha,$
inequalities (11) to (14), and set $S_1(j) = S_1$,
$X(j) = X, \epsilon_{inv}(j) = \epsilon, \epsilon_{inv}(j) = \epsilon_{inv}, \tau(j) = \tau,$
$\tau_{inv}, \rho(j) = \rho, \rho_{inv}(j) = \rho_{inv}.$

2.3 If α satisfies $2n + 6 < \alpha < 2n + 6 + \kappa_1$ for some small κ_1, then go to Step 3 with $\overline{\gamma}(k) = \gamma(k) - \delta_\gamma$ where $\delta_\gamma > 0$ is some small constant, else go to Step 2.4.

2.4 If $j < N$ where $N > 0$ is some large integer, then go to Step 2.2, else go to Step 4 with $\gamma = \gamma(k)$.

3 Design a guaranteed cost controller with a full order observer under a fixed S_2 for $\overline{\gamma}(k)$, when k = 1, 2, ···.

3.1 Solve a convex problem

$\begin{bmatrix} S_1 & I \\ I & X \end{bmatrix} > 0, \gamma_1 + \gamma_2 < \gamma(k), \epsilon\epsilon_{inv} > 0,$

$\tau\tau_{inv} > 0, \rho\rho_{inv} > 0$, inequalities (11) to (14),

If infeasible, then go to Step 4 with $\gamma = \gamma(k)$, else set $S_1(0) = S_1, X(0) = X, \epsilon(0) = \epsilon, \epsilon_{inv}(0) = \epsilon_{inv},$
$\tau(0) = \tau, \tau_{inv}(0) = \tau_{inv}, \rho(0) = \rho, \rho_{inv}(0) = \rho_{inv}.$

3.2 When j = 1, 2, ···, solve a minimisation problem of α by LMI under

$\begin{bmatrix} S_1 & I \\ I & X \end{bmatrix} > 0, \gamma_1 + \gamma_2 < \gamma(k), \epsilon\epsilon_{inv} > 0,$

$\tau\tau_{inv} > 0, \rho\rho_{inv} > 0,$

$\mathrm{tr}[S_1(j)X + X(j)S_1] + \epsilon(j)\epsilon_{inv} + \epsilon\epsilon_{inv}(j) + \ldots$
$\tau(j)\tau_{inv} + \tau\tau_{inv}(j) + \rho(j)\rho_{inv} + \rho\rho_{inv}(j) < \alpha,$
inequalities (11) to (14), and set $S_1(j) = S_1$,
$X(j) = X, \epsilon_{inv}(j) = \epsilon, \epsilon_{inv}(j) = \epsilon_{inv}, \tau(j) = \tau,$
$\tau_{inv}(j) = \tau_{inv}, \rho(j) = \rho, \rho_{inv}(j) = \rho_{inv}.$

3.3 If α satisfies $2n + 6 < \alpha < 2n + 6 + \kappa_2$ for some small κ_2, then go to Step 2 with $\gamma(k+1) = \overline{\gamma}(k) - \delta_\gamma$, else go to Step 3.4.

3.4 If $j < N$, then go to Step 3.2, else go to Step 4 with $\gamma = \overline{\gamma}(k)$.

4 End with $S_1 = S_1(j), S_2, X = X(j), \epsilon = \epsilon(j), \epsilon_{inv} = \epsilon(j),$ $\tau = \tau(j), \tau_{inv} = \tau_{inv}(j), \rho = \rho(j), \rho_{inv} = \rho_{inv}(j), \overline{L}, \gamma,$ $\gamma_1, \gamma2$.

Note that N in Steps 2.4 and 3.4 is the threshold to decide the solvability of the minimisation problems. Namely, this means that there is no solution satisfying $\gamma(k)$ or $\overline{\gamma}(k)$. Note also that $\gamma(1)$ is needed to choose the value such that the algorithm can go to Step 3. We determine a sufficiently large $\gamma(1)$ such that there exists a feasible solution. To simplify, we use a twice of optimal guaranteed cost value obtained in the design of a minimal order observer-based guaranteed cost controller.

4 An illustrative example

This section verifies the proposed method by providing a numerical example solved by LMI control toolbox of MATLAB. Consider a system (Inoue, 1977) with

$$A = \begin{bmatrix} -3 & 0 & -2 & 0 \\ 0 & -2 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 3 \\ 2 \\ -6 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} O_2 & I_2 \end{bmatrix}, \boldsymbol{m}_0 = \boldsymbol{0}_4, \Sigma_0 = I_4, R = 9,$$

$$Q = diag(7, 15, 1, 3), D_A = \begin{bmatrix} 0.1I_2 & O_2 \\ O_2 & O_2 \end{bmatrix},$$

$$E_A = \begin{bmatrix} 0.3I_2 & 0.3I_2 \\ O_2 & O_2 \end{bmatrix}, \Lambda = \begin{bmatrix} -50 & 0 \\ 0 & -30 \end{bmatrix},$$

$$D_B = diag(0.3, 0.1, 0.3, 0.1), E_B = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

Applying Theorem 1 with $\gamma(1)$ = 33.7786, we obtain a solution

$$S_1 = \begin{bmatrix} 1.1251 & 0.1384 & 0.0591 & 0.1487 \\ 0.1384 & 4.6164 & 1.2272 & 1.7050 \\ 0.0591 & 1.2272 & 1.4415 & 1.3071 \\ 0.1487 & 1.7050 & 1.3071 & 8.2134 \end{bmatrix},$$

$$S_2 = \begin{bmatrix} 0.4520 & 0.6661 & -0.1467 & -0.0592 \\ 0.6661 & 1.0182 & -0.2013 & -0.1254 \\ -0.1467 & -0.2013 & 0.0764 & -0.0288 \\ -0.0592 & -0.1254 & -0.0288 & 0.0973 \end{bmatrix},$$

$$X = \begin{bmatrix} 0.8929 & -0.0203 & -0.0099 & -0.0104 \\ -0.0203 & 0.2844 & -0.2198 & -0.0237 \\ -0.0099 & -0.2198 & 0.9813 & -0.1104 \\ -0.0104 & -0.0237 & -0.1104 & 0.1444 \end{bmatrix},$$

$$\epsilon = 4.9249,\ \epsilon_{inv} = 0.2030,\ \rho = 0.9552,$$

$$\rho_{inv} = 1.0469,\ \tau = 4.3247,\ \tau_{inv} = 0.2312$$

$$\bar{L} = \begin{bmatrix} 0.1840 & 0.0433 \\ 0.0070 & 0.2483 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$K = \begin{bmatrix} -0.3830 & -0.4433 & 0.5234 & -0.4697 \end{bmatrix},$$

$$\gamma = E\left[J^*\right] = 17.0408.$$

All closed-loop poles are calculated for $F_A = F_B = I$ as –5.4457, –0.9828, –1.000, and –3.4451. For the initial values of the system state variables and the observer state variables are set as $\boldsymbol{x}(0) = [-1\ 2\ 3\ -2]^T$ and $\hat{\boldsymbol{x}}(0) = [0\,0\,0\,0]^T$, simulation results are shown in Figures 1 to 4. The trajectories converge to the origin. Figure 5 depicts the trajectory of the guaranteed cost γ.

Figure 1 Trajectories of states $x_1(\cdots)$, $x_2(—)$, $x_3(- - -)$ and $x_4(- -)$

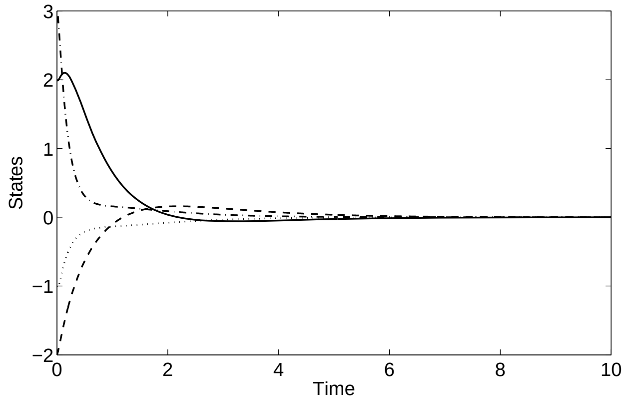


Figure 2 Trajectories of errors $e_1(\cdots)$, $e_2(—)$, $e_3(- - -)$ and $e_4(- -)$

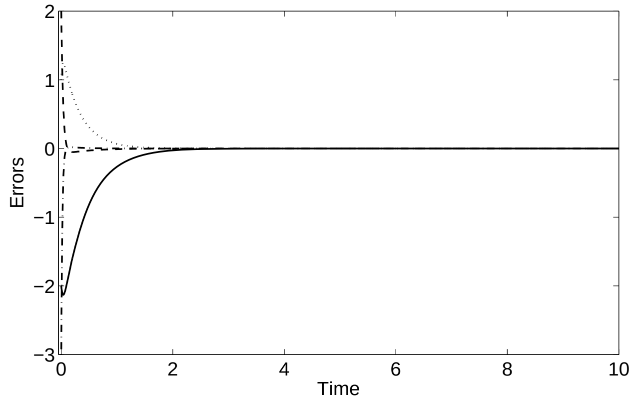


Figure 3 Trajectories of outputs $y_1(—)$ and $y_2(\cdot\cdot\cdot)$

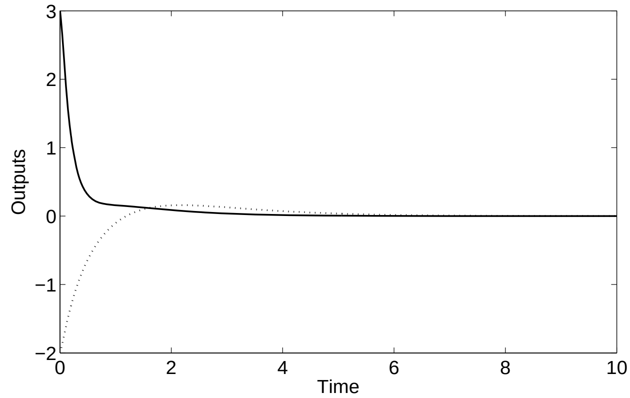


Figure 4 Trajectory of input

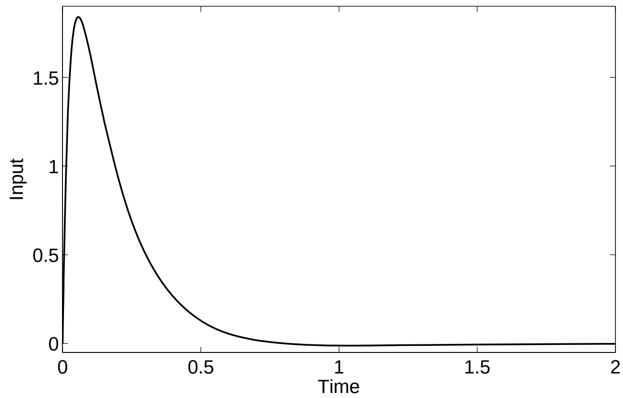
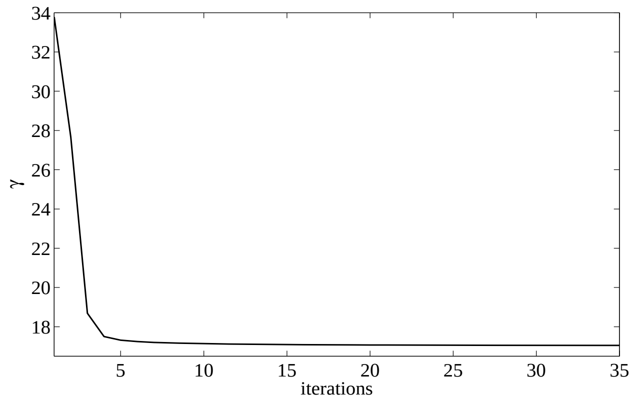


Figure 5 Trajectory of γ



5 Conclusions

This paper discusses a full order observer-based guaranteed cost control problem. A sufficient condition for the existence of state feedback guaranteed cost controllers is derived to determine the stability on the LMIs term. A numerical example with simulation results is given to illustrate the proposed method.

References

Boyd, S., Ghaoui, L.E., Feron, E. and Balakrishnan, V. (1994) *Linear Matrix Inequalities in System and Control Theory*, SIAM, Philadelphia.

Cao, Y., Lam, J. and Sun, Y. (1998) 'Static output feedback stabilization: an ILMI approach', *Automatica*, Vol. 34, No. 12, pp.1641–1645.

Chang, S.S.L. and Peng, T.K.C. (1972) 'Adaptive guaranteed cost control of systems with uncertain parameters', *IEEE Trans. on Automatic Control*, Vol. 17, No. 4, pp.474–483.

Chen, W., Guan, Z. and Lu, X. (2004) 'Delay-dependent output feedback guaranteed cost control for uncertain time-delay systems', *Automatica*, Vol. 40, No. 7, pp.1263–1268.

Esfahani, S.H. and Petersen, I.R. (1998) 'An LMI approach to the output-feedback guaranteed cost control for uncertain time-delay systems', *Proc. of the 37th IEEE Conference on Decision and Control*, pp.1358–1363.

Ghaoui, E., Oustry, F. and AitRami, M. (1997) 'A cone complementarity linearization algorithm for static output-feedback and related problems', *IEEE Trans. on Automatic Control*, Vol. 42, No. 8, pp.1171–1176.

Inoue, A. (1977) 'Studies on the designs of observers for linear control systems', PhD thesis, Kyoto University.

Ishitobi, M. and Miyachi, M. (2008) 'An ILMI approach to guaranteed cost controllers with an observer', *Dynamics of Continuous, Discrete and Impulsive Systems B*, Vol. 15, pp.317–326.

Leibfritz, F. (2001) 'An LMI-based algorithm for designing suboptimal static H_2/H_∞ output feedback controllers', *SIAM Journal of Control Optimization*, Vol. 39, No. 6, pp.1711–1735.

Lien, C.H. (2004) 'Robust observer-based control of systems with state perturbations via LMI approach', *IEEE Trans. on Automatic Control*, Vol. 49, No. 8, pp.1365–1370.

Lien, C.H. (2005) 'Guaranteed cost observer-based controls for a class of uncertain neutral time-delay systems', *Journal of Optimization Theory and Applications*, Vol. 126, No. 1, pp.137–156.

Mahmoud, M.S. and Zribi, M. (2003) 'Guaranteed cost observer-based control of uncertain time-lag systems', *Computers and Electrical Engineering*, Vol. 29, No. 1, pp.193–212.

Matsunaga, D., Ishitobi, M. and Kunimatsu, S. (2009) 'An ILMI approach to guaranteed cost controllers with a minimal order observer', *Proc. of ICROS-SICE International Joint Conference 2009*, pp.583–588.

Matsuo, T., Adachi, K. and Suemitsu, H. (2008) 'Frequency estimation with LMI-based adaptive update law', *International Journal of Advanced Mechatronic Systems*, Vol. 1, No. 2, pp.100–106.

Miller, R.A. (1973) 'Specific optimal control of the linear regulator using a minimal order observer', *Int. J. Control*, Vol. 18, No. 1, pp.139–159.

Petersen, I.R. and McFarlane, D.C. (1994) 'Optimal quadratic guaranteed cost control and filtering for uncertain linear systems', *IEEE Trans. on Automatic Control*, Vol. 39, No. 9, pp.1971–1977.

Sato, K., Nakashima, T. and Tsuruta, K. (2008) 'A robust adaptive H_∞ control for robotic manipulators with input torque uncertainties', *International Journal of Advanced Mechatronic Systems*, Vol. 1, No. 2, pp.116–124.

Telford, A.J. and Moore, J.B. (1989) 'Doubly coprime factorizations, reduced order observers, and dynamic state estimate feedback', *Int. J. Control*, Vol. 50, No. 6, pp.2583–2597.

Won, S. and Park, J.H. (1999) 'Design of observer-based controller for perturbed time-delay systems', *JSME International Journal C*, Vol. 42, No. 1, pp.129–132.

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